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THE MATHEMATICS TEACHER

VOLUME XVI APRIL, 1923 NUMBER 4

PSYCHOLOGICAL TESTS OF MATHEMATICAL ABILITY AND EDUCATIONAL GUIDANCE

By PROFESSOR AGNES L. ROGERS
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One of the most noteworthy features of modern times is the increased recognition of the need for educational guidance and the application of scientific method in the means used to meet it. This new trend can be observed at several levels of education. It is not more conspicuous in high school than in college, where excellent work is being done to ensure that the gifts of college youth are not wasted or misapplied in directions in which the individual cannot be reasonably expected to attain success and satisfaction. Notable developments of the past year or so are the plans devised to guide the college student into the right channel of activity.

For this purpose it should be self-evident that tests of general intelligence do not suffice. In proportion as they succeed in measuring all-round capacity, to that degree they fail to indicate the special lines of strength and weakness that are characteristic. The object of an intelligence test is to guage as rich a variety of abilities as possible and to combine the results, so that if an individual is inferior in certain directions, his superiority in others will compensate in the total score and the final summation will roughly indicate his general power to cope with the majority of situations that life may present.

For guidance after admission to an institution or grade requiring a certain level of capacity, a more diagnostic instrument is required. Educational advisers feel the need to know whether a particular student excels along mathematical, linguistic or artistic lines. To meet this requirement use is now being made of "placement" examinations in distinction from intelligence tests. These examinations are tests of achievement. They measure the results of past experience and training, but they are designed to gauge the power of the student to utilize previous experience and achieved knowledge rather than merely to recall it. As I urged in 1919, in considering the desirable types of

college entrance examination it should first be decided what abilities are crucial for success in the various kinds of specialized work a college course demands, then measuring instruments of these specific abilities should be constructed and finally satisfactory standards should be determined. The program of work this implies is already begun at Columbia College and promises to give results of great value.

The problem of educational guidance in high school assumes a somewhat different form. As it presents itself to the public high school principal it is much more difficult to solve. In general the principal of a public high school holds one criterion of his success to be the extent to which the school retains pupils throughout the high school course. He considers an excessive elimination as tantamount to the failure of the high school to serve the community as it should. The present situation from this point of view is unsatisfactory since the percentage of students who do not complete the course over the country at large ranges from 50 to 85. These figures indicate the magnitude of the problem, but they do not tell the whole story. In addition to the feature of elimination, there is that of transfer and the number of transfers in high schools where pupils must choose a definite course—classical, commercial and so forth—is extremely large. So wasteful has this appeared that widespread use is being made of intelligence tests, apparently to prevent the choice of courses that are too difficult, but also to adjust the amount of material and the rate of learning to capacity. It is contended that in certain cities three-fourths of the failures in the first year in technical and English courses have been prevented by careful classification and adapting standards and teaching methods to the powers of the pupils. Whereas in the past the policy, as far as high school education is concerned, has been survival of the fittest and levelling work down to the ability of the average student, it is today felt to be desirable to give each individual all the work he can take as fast as he can take it.

The discovery of what he can take is as serious a task for the high school as the college and the problem presents peculiar difficulties. On entering college the student has already sampled many kinds of intellectual experience. A measure of the results of his efforts after some years of study is as fair a clue to his capacity to master learning in any subject as any that the wit

of man can devise. We have scientific evidence that school marks, faulty though they are, if they are sufficient in amount, and include the records of several instructors and if properly weighted give a reasonably accurate measure of future academic success. In fields where experience has not yet been had or where it has been limited in character, especially where initial experience bears no very close relationship to future experience in the science or study involved, the difficulty of forecasting whether it will be worth while to continue such study is obviously great and we cannot use school marks with confidence.

Thus the high school being the stage at which the pupil embarks on a differentiated curriculum presents something in the nature of a conundrum in educational guidance. Here, if anywhere, expert direction is required. Here, if anywhere, false steps are costly to the pupil and at this stage "placement" examinations, in other words standardized tests of achievement are at certain points when choice confronts, either not available or available in so scant a measure and so fallible a form as to be inadequate for the purpose.

If a high school student has not succeeded with mathematical work after a year's experience with one or more of the mathematical subjects, several reasons may be given to explain his failure. He may possibly lack to a serious extent the ability to think quantitatively or in terms of spatial relations. On the other hand he may possess the requisite abilities in sufficient amount to go on to higher work and his failure may be due principally to poor teaching and the consequent irksome monotony of the first year's work in mathematics. Again, as is too often the case unfortunately, he may have mathematical capacity but be deficient in perseverance and other moral qualities which are indispensable for success in any field of intellectual work. The core of the difficulty is that we need to know in advance of their exercise whether a pupil has the powers demanded by subsequent mathematical work. It is a paradox but the essence of the problem that we require an instrument to foretell in advance of their use whether a pupil has the powers that will make it profitable for him to study geometry, algebra and trigonometry. It was with the purpose of solving this problem at the high school level I essayed an analysis of mathematical ability and the construction of a battery of tests which should stress power

rather than memory, and quick analysis of new mathematical situations rather than repetition of established responses in a familiar field.

We have to admit that our test technique for determining the probability of success in any field is imperfect. It is, however, becoming more satisfactory. Improvements have been made since the publication of the original study of mathematical ability which was made by the writer in 1917. Perhaps it also requires to be said even to a group of teachers of mathematics that predictions can never be absolute, and that mistakes will certainly be made. All we can do is to state whether a pupil's chances of succeeding are small or great. Thus if a student takes the tests of mathematical ability and obtains a very low score, we cannot assert that absolute failure confronts him; we can, however, affirm that his likelihood of success is small and that we have good warrant for the statement. While prognostic tests are imperfect, teachers' judgments and school marks also err and err conspicuously in forecasting accomplishment in geometry on the basis of the pupil's achievement in algebra or arithmetic. Thus tests as prognostic tools, though far from ideal, are usually superior to other available means of determining the chances of success with the course of study in mathematics in the senior high school, and can serve as a useful check upon other information available.

The investigation which led to the construction of the sextet of tests of mathematical ability was for the most part an essay in method. Although it was given originally to small numbers and to girls only, it was applied in very carefully controlled conditions, where all the facts were known and it is surely more scientific to base conclusions on careful work on two groups of lesser numbers than on larger groups of which little is directly or definitely known, and where those applying the tests may possibly be poorly equipped for the work.

On this occasion, however, I am to present the results accumulated in the past three years which have been derived from the new form of the test and where the examiners have been high school teachers testing their own pupils. The writer is greatly indebted for their cooperation. The sextet has been published in a new form since March, 1921. Largely owing to the help and initiative of Professor C. B. Upton, several improvements

have been effected. It is now in handy booklet-form. Directions have been simplified and carefully recast to avoid ambiguities. Scoring has been made more easy by means of helpful score cards. Time limits have in one or two tests been shortened, so that the total time for the six tests is now sixty minutes. The changes in content made are negligible.

Three topics will be dwelt on in this report:

(1) First, data will be presented to show anew the prognostic power of the test.

(2) Second, data will be presented to indicate the superiority of the tests to tests of general intelligence for sectioning students in mathematical classes on the basis of intellectual ability.

(3) Third, a discussion of standards will be given.

The predictive power of the sextet of tests of mathematical ability was determined in the original investigation published under the title "Tests of Mathematical Ability and Their Prognostic Value" by showing the amount of correspondence that was found between the total scores obtained and subsequent school marks in mathematics. In the case of the Horace Mann School this correspondence amounted to .82 ± .03 as measured by the Pearson coefficient of correlation. The fact that complete agreement between tests and school marks was not obtained was attributed to various factors, such as the unreliability of school marks, flaws in the tests and also to the exercise in the tests of mental capacities not covered by school marks, though requisite in more advanced mathematical work. The criterion to determine the worth of the sextet that will now be used demands attention. It is important to note that it varies, in some cases including school marks for algebra for one-half year, or for one complete academic year or school marks for varying periods of study of algebra and geometry. Table I presents the coefficient of correlation found between the total test scores and varying amounts of school marks. The standard Bravais-Pearson method was used. Coefficients obtained range from as low as .34 to .76.

It is noteworthy (1) that all the criteria available for evaluating the tests are limited. The school marks furnished by those using the tests vary from grades for only one semester in algebra to a year of algebra and a semester of geometry. (2) The coefficients obtained are smaller than those obtained in the

original investigation. This may be due to chance factors, but probably experience in testing was an influence. It was noted for example, in the case of some returns that zero scores were included for some tests. This should never occur with adequate

TABLE I

CORRELATION BETWEEN TOTAL TEST SCORE AND VARYING OF SCHOOL MARKS

	School	No. Tested	Mathematics Studied	Correlations
A. School Marks for ½ year Algebra:	Batavia, Ill.	102	1 semester algebra	r — .56 ± .05
B. School Marks for 1 year Algebra:	Joliet, Ill.	26	2 semesters algebra	r — .48 ± .10
	Newton, Mass.	31		r — .70 ± .06
C. School Marks for 1 year Algebra; 4 months Geometry:	Belmont, Mass.	36	1 year algerbra 4 months geometry	r — .34 ± .10
	Ord, Neb.	50	1 year algebra ½ year geometry	r — .42 ± .08
	Huntingdon, Va.	116	1 year algebra ½ year geometry	r — .41 ± .05
	Ypsilanti, Mich.	32	2 semesters algebra 1 semester geometry	r — .76 ±
D. School Marks for ½ year Algebra; ½ year Geometry:	Horace Mann School for Girls, N.Y.	46	½ year algebra ½ year geometry	r — .75 ± .04
E. School Marks for 1 year Geometry:	Western High School Baltimore, Md.	38	1 year algebra, 1 month geometry	r — .71 ± .05
	Western High School Baltimore, Md.	37	1 year algebra, 1 month geometry	r — .56 ± .07

administration of these tests. (3) In the third place the coefficients are notably higher where the test conditions are known to have been carefully conducted, for instance, at the Horace Mann School, New York, and at Western High School, Baltimore. The coefficients in the case of the Baltimore results are relatively large considering that the school marks include only a year's grades in plane geometry. The coefficient obtained between scores on the first test in the sextet—the geometry test alone, and school marks in geometry amounted to .62 for one group and .44 for the other.

A satisfactory coefficient of correlation between a reliable criterion of success and a prognostic test, where the purpose in view is placement of students in groups of equal ability should be greater than seven-tenths. In the original investigation a coefficient of over eight-tenths was obtained. It would be better to exceed that amount. The chief reason for presenting these results is to enlist the efforts of teachers of mathematics to apply the sexet and to keep careful records of the success with school work of the students who have been tested, and particularly to use such objective measures of mathematical *achievement* as are now available, which were not yet constructed when the original investigation was made. (We have now many tests of algebraic attainment, for example, Hotz's tests, Rugg and Clark's and Monroe's. We have Schorling's geometry test for measuring attainment in geometrical work and Minnick's admirable tests.) That there is significant correspondence between success with the tests and success with school work in mathematics is evident. We need, however, a more comprehensive investigation covering a longer period. Where the criteria for evaluating the sextet are so inadequate, it is impossible to reach a clear-cut decision as to the practical worth of the tests. The low coefficients merely warn us against relying on the tests alone or school marks alone. Both undoubtedly are indices to the pupil's mathematical ability. In proportion to the time spent the tests certainly have real advantages.

The question has frequently been raised by mathematics teachers whether an intelligence test would not answer our needs as far as admission to or classification of mathematics classes is concerned. We have already stated our objection to this point of view in general, but we would also call attention to two

studies. The first was made by Miss L. Dixon of Western High School, Baltimore, with the coöperation of the writer. Tests were applied by the writer to two classes in the second year of the senior high school. In one group there were 41 and in the other 43 students respectively. Both classes had had ten months of formal algebra and approximately one month of intuitional geometry. At the end of the year the coefficients of correlation between the mathematical ability tests and school marks in Geometry, English and French were determined by two groups. They were as shown in Table II.

TABLE II

Coefficients of Correlation between Mathematical Ability Sextet and School Marks for Various Subjects (Western High School, Baltimore, Maryland):

	Group B 5	Group B 6
Mathematical Ability Sextet and School Marks in Geometry	$r = .71 \pm .05$	$r = .56 \pm .07$
Mathematical Ability Sextet and School Marks in English	$r = .46 \pm .08$	$r = .18 \pm .10$
Mathematical Ability Sextet and School Marks in French	$r = .54 \pm .07$	$r = .15 \pm .10$

The two groups differed in character, group B-5 being recognized to be a superior group. The two should therefore be considered separately. The difference between the correspondences found in the case of geometry alone and that found in the case of English or French is not extremely great, but it is in the right direction and if we bear in mind that geometry is only one phase of mathematical capacity we are probably justified in asserting on the basis of these results that the tests do succeed in locating elements in ability, specific and peculiar to mathematical intelligence in distinction from general intelligence.

The second study reinforcing these results is the report of the application of the sextet together with the Otis intelligence examination in Champaign High School, by W. S. Monroe.[1] Thirty-

[1] Monroe, Walter S. Some correlations between Otis Scale and Rogers Mathematical Tests, Journal of Educational Research: II, 4, November, 1920, 774-776.

nine students, all of whom had failed in mathematics were given these two tests. Their intelligence level is indicated by their coefficients of brightness, which were distributed as follows:

TABLE III

Coefficients of Brightness

65-74	75-84	85-94	95-104	105-114	115-124	125-134	135-144
2	2	6	9	9	7	3	1

It will be noted that on the whole they are above the average in general ability. The correlation found between the Otis scale and the mathematical ability tests was .41 ± .09. The correlations with the individual tests in the sextet were as follows:

TABLE IV

Otis	with	1. Geometry	.17 ± .12
"	"	2. Algebraic Computation	.37 ± .10
"	"	3. Interpolation	.58 ± .08
"	"	4. Superposition	.02 ± .12
"	"	5. Trabue Language Scales L. & M.	.52 ± .09
"	"	6. Mixed Relations	.47 ± .09

Obviously these two instruments have certain features in common, features which would explain in part the interdependence found. For instance, both demand in some measure the ability to follow directions, both have tests with geometrical material, both have tests of verbal ability—the sextet having language completion and analogies and the Otis intelligence examination having disarranged sentences and finally the former has arithmetical problems which certainly tap one phase of mathematical capacity.

The intercorrelations between tests involving similar activities one would expect to be substantial. Monroe cites the following interesting figures. The correlation between the Rogers Sextet and the Otis Arithmetic test and the Otis Geometrical Figures test combined amounted to .54 ± .08. The correlation between the Rogers Sextet and the Otis Arithmetic test alone was .53 ± .08 and with the Otis Geometry test alone was .41 ± .09. The correlation obtained between the Rogers Geometry test (test 1 of the sextet) and the Otis test on Geometrical Figures (test 6 of the Otis Scale) was .35 ± .10.

The fact is that the makers of so-called intelligence tests looked around for handy tests easy to give and score and as their work-

ing principle was "test as many abilities as possible and pool the results" and as tests of the abilities involved in quantitative thinking and thinking in space relations happened to be well developed, they took them and incorporated them with other material available. Material for testing other phases of intelligence was not so easy to procure and hence so-called intelligence tests measure in general too narrow a range of powers. Again, in our analysis of mathematical ability we found the ability to use the veracular important, hence we incorporated existing tests of verbal ability. These facts explain the correspondence that is found in the study reported. It is very significant that there is no correspondence between the intelligence test results and the results of the tests of geometrical abilities. This suggests that if a good prognostic test for mathematical capacity is developed, the divergence between it and tests of general capacity will be substantial. It is apparent from the above figures that the tests of geometrical ability fail to be measured by tests of general ability and were our tests of algebraic ability satisfactory, the same would substantially hold true of them. We need a new series of mathematical tests requiring innate talent; such tests, for example, as the finding of the nth term in algebra and determining loci in geometry are the kind of tests in question. The abilities involved in these are intuitive and indispensable for later work and therefore likely to prove prognostic to a higher degree than the tests in the present sextet. It would be a step in advance to make a new study along these lines.

The interpretation of the final score obtained by a pupil depends upon the standards that have been found satisfactory up to the present. Certain general principles have to be kept in mind in this connection. Chapman[1] has pointed out that in determining norms of achievement for educational tests it has been a common practice to combine the results from groups which are unhomogeneous and to obtain norms which are practically meaningless. The value of a norm is proportionate to the care with which the group tested has been selected and the degree of accurate knowledge which we have about it. He points out that norms based on tests given in different months of the year, to grades named identically, but in fact differing greatly, cannot

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[1] Chapman, J. C. Some Elementary Statistical Considerations in Educational Measurements, Journal of Educational Research; IV, 3, 212-220.

throw much light on other scientific situations. Much more illuminating would it be to keep the distributions separate, stating specifically in each case the date of application of the tests and the character of the group tested. Undoubtedly grades similarly named vary even in the same school system, still more are they likely to vary in systems in different localities.

The desirability of keeping results based on different conditions distinct holds also for norms whose purpose is prognosis. A variety of practices exists in the amount and character of the mathematical training which a pupil may have had by the time he has reached the ninth or tenth school year. Consequently we present what seem to us typical results selected from investigations including twelve schools and one thousand, two hundred and seventy-eight pupils.

Type A

Table V gives the distribution of scores obtained in *September* by *204* students at the beginning of the tenth school year who have had one year of formal algebra and half a year of geometry.

TABLE V

	Percent.
Under 200	1
200-299	14
300-399	36
400-499	32
500-599	13
600-699	4
700-799	½
800	0

Table VI gives the distribution of scores obtained in *January* by a more highly selected group of *38* students who have had one year of formal algebra and four months of demonstrational geometry.

TABLE VI

	Percent.
Under 200	0
200-299	3
300-399	5
400-499	26
500-599	48
600-699	11
700-799	8
800-899	0

Type B

Table VII gives the distribution of scores obtained in *June* by *85* students who had had one year of formal algebra.

TABLE VII

	Percent
Under 200	4
200-299	7
300-399	26
400-499	35
500-599	16
600-699	8
700-799	1
800	

Type C

Table VIII gives the distribution of scores obtained in *June* by *102* students who had had less than one-half year of formal algebra.

TABLE VIII

	Percent
Under 200	10
200-299	35
300-399	28
400-499	23
500-599	6
600	0
700	0
800	0

Table IX presents these results in the form of a percentile table.

TABLE IX

Percentile Table for Different Types of Schools

	10	20	30	40	50	60	70	80	90
Type A-1.	270	315	345	374	398	422	462	492	525
Type A-2.	408	456	492	509	524	539	566	596	664
Type B	275	347	384	412	444	467	493	547	625
Type C	181	233	282	295	326	360	389	433	486

These results indicate that there are marked differences in results in different schools. They also indicate as we would expect that the scores in general on the tests of mathematical ability are affected by mathematical training. This has a less marked effect, however, than we would customarily expect. Thus, of 85 pupils having had one year's instruction in algebra in one school ten and one-half per cent had scores under 300, whereas fifteen per cent of 204 pupils in another school in a different locality obtained scores less than 300.

From these facts it should be clear that the tests, as has to be maintained of all such predictive instruments whether for general or special capacity, should not be used independently but only in conjunction with other valuable evidence of the pupil's powers, namely teachers' judgments or school marks. It is the agreement of these independent indices of the pupil's ability that justifies us in taking action and recommending continuance or discontinuance of the subject.

The standards we had already laid down seem to us to be somewhat higher than they should be in the light of returns made to us. We, therefore, recommend on the basis of our results from the application of the sextet by classroom teachers, that a score over 650 indicates very superior talent, a score over 550 indicates superior capacity, 350-550 indicates average capacity, 250-350 inferior ability and under 250 very inferior power. The first two groups should be offered an opportunity of covering the ground more rapidly, the 250-350 group should be given a longer time to accomplish the same amount of work than the middle group, while those securing less than 250 should be released from further study of the subject save in its historical aspects.

TEXTBOOKS IN UNIFIED MATHEMATICS FOR COLLEGE FRESHMEN

By VERA SANFORD
The Lincoln School

In recent years, many colleges have altered their courses in freshman mathematics with the object, as one teacher expresses it, of "condensation and acceleration": the condensation is effected by the elimination of subject matter whose intrinsic usefulness is slight; the acceleration consists in an earlier approach to important topics which, under the old regime, would be postponed to a later year when the student might or might not elect them.

Often these changes leave the various sections of the work in their old compartments of trigonometry, analytics, and so on. But with the shortening of each division by the omission of material whose importance is relatively small, time is gained for the addition of work in the calculus. This scheme lends itself particularly well to the college that is obliged to have its freshman course split into two or more distinct parts.

Other colleges have broken down the compartment idea. The acceleration through condensation can thus be gained in more efficient form, for many parts of the subject matter profit by the introduction of topics that would ordinarily come later in the course. This form of organization also offers space for things not ordinarily included in the old time-honored divisions as, for instance, mathematics of finance, elementary statistics, work with empirical data, and so on. One must not conclude, however, that all such courses contain this specific material.

The two forms suggest the contrast between the two types of junior high school mathematics: the one where the work is now arithmetic, now algebra, now geometry; and the other where, in his study of general mathematics, the child is scarcely conscious that a grownup would call one part of his lesson trigonometry and another algebra. The second type presents the same danger in college that it does in the seventh, eighth and ninth grades, namely that the student will become a "jack of all trades but master of none." In both cases, the responsibility is divided between the people who write the texts and the people who teach the classes.

To the teacher of senior high school mathematics, the second type of reorganization of the freshman college courses is a more interesting field than the first or telescoped one. Possibly this is because we were exposed to the stratified system ourselves, and so we hail a novel thing that appears to avoid the drawbacks of the one we knew. Perhaps if we had taken a unified course, we might now wish we had had one in which you first studied trigonometry, then analytics and so forth, beginning each subject with its alphabet and progressing by logical steps to words of n-syllables. But, as I understand it, the colleges that offer the unified work for freshmen, give ample opportunity for intensive work in the different branches of mathematics in subsequent courses.

An acquaintance with these freshman courses is desirable on several grounds: it is of interest to know what different types of work the students who go to college will meet; it is probable that these courses contain valuable hints as to ways in which the various topics which we treat in the senior high school may be made more vital; but, most important of all, as people interested in mathematics, it is essential for us to realize that, as a child may learn to read without first learning the alphabet, so a student of mathematics may gain real power in the calculus without first studying such things as the focal chord and latus rectum of a parabola. We are too prone to think that the road we traveled is the only one, and to feel that the things we did constitute the *sine qua non* of sound work in more advanced mathematics. This last objective, i. e. the realization of the relatively small amount of analytics et al needed to bridge the gap between the high school mathematics and the calculus, can best be gained by working back from some important problem that demands a knowledge of the calculus, keeping note of the work in the other subjects that this entails. This would give an idea of the topics one would need to put into a course of minimum essentials. But there is a wide discrepancy between solving a single problem and giving a student a working knowledge of the subject. A comparison of one of the newer textbooks for freshmen with the Analytics and College Algebra we used ourselves, gives a striking illustration of the tremendous difference in content and approach between the two. Both the topical and the unified schemes of reorganization are valuable

in this regard, but I think that no one can fail to grant that the second type is the more fruitful field for the person who is concerned with the problems of the senior high school course. In a self-satisfied way, we may have anticipated that the changes in freshman college mathematics would follow conservatively upon the changes in the senior high school work, and when we realize that these changes are actually taking place in advance of those in the senior high school, it surely behooves us to know what is going on. We certainly wish to know what aims prompt the reorganization, and what material is being selected to meet these aims.

The object of this paper, then, is to investigate the purpose and content of these unified courses through the textbooks now available. Eleven[1] of these have been examined, and although this list is not exhaustive of the textbooks now in print, and although it does not cover the courses where the work has not yet reached the printed stage, these eleven books may be assumed to give a fair notion of the divergence among the various reorganizations of freshman mathematics, and we may infer that they also give an estimate of the topics common to the group.

Like the junior high school series to which they are close contemporaries, these books show strong similarity of purpose (wherever the author takes us into his confidence), but even in comparison with the junior high school books, they exhibit wide variation of content.

The prefaces show two distinct motives, practical and cultural. The *practical* is not in the way of manipulation, but in giving a comprehension of the principles that are important in later work

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[1] Smith and Granville, Elementary Analysis, Ginn and Co., 1910.
Slichter, Elementary Mathematical Analysis, McGraw Hill Book Co., 1914, 1918.
Young and Morgan, Elementary Mathematical Analysis, Macmillan, 1917.
Karpinski, Benedict, and Calhoun, Unified Mathematics, D. C. Heath and Co., 1918.
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Griffin, An Introduction to Mathematical Analysis, Houghton Mifflin Company, 1921.
I am indebted to Professor J. W. Young for this bibliography. To it should be added Longley and Wilson, Freshman Mathematics, Whitlock's Book Store, Inc., New Haven. The edition of this text is temporarily exhausted so a copy could not be obtained for this study.

in science, to save time in reaching the vantage ground of the calculus, and to enable the student to express the laws underlying empirical data of simple types. The *cultural* aim is phrased in different ways: "to give a deeper understanding of the significance of mathematical principles and relations,"[1] "to give a conception of what mathematics has done and is doing for mankind,"[2] "to emphasize the essential harmony and interplay between the two great fields . . . analysis and geometry" . . . and to show mathematics as a "powerful tool of science, playing a wonderful part in the development of civilization."[3] It is noted by one of these authors that the two aims are not contradictory. The broader field needed for the practical purposes lends itself under right teaching, to the cultural purpose as well. Another author maintains that this work is not only better than the traditional course for the student who drops his mathematics at the end of his freshman year, but that with a properly arranged sophomore course, the student of mathematics loses no time. If one may make a guess, he has a far better comprehension of his work than he would have had under the older course.

The range of subject matter is large, and several authors provide supplementary material for classes whose preliminary work has included subjects treated in the course. One book assumes trigonometry. Another presupposes so little training in mathematics that the author states that it is possible for a person who has had but two years of secondary school mathematics to do the work. Most of them offer a review of algebra, either in preliminary chapters or in an appendix. In the main these reviews are good, but one is sometimes puzzled to know whether the work following will offer occasion to display one's ability to juggle the complicated fractions and radical signs he has just studied.

On the score of the calculus, these texts present all shades of emphasis: no calculus at all, the use of graphs to solve maxima and minima problems, the notion of a derivative introduced

[1] McClenon.

[2] Young and Morgan.

[3] Karpinski.

Of these eleven books, eight specifically mention the practical aim, five the cultural one, four state that the unifying thread of their work is the function concept. Others show in their tables of contents and in their treatment of the subject that their work is built on this also.

without its awe-inspiring name, a small amount of differential calculus, and a considerable unit. Five of the eleven give integral calculus as well as differential.

The diversities in subject matter among the texts are best shown by a comparative table, and in view of the object of these courses—to give the student the mathematics he will need in future work in the sciences—it may not be illogical to check against each the items given in the investigation of the National Committee[1] as to the relative importance of the various topics in preparation for the work of the elementary college courses in the physical and social sciences. For although the texts represent the independent work of diverse groups, the investigation mentioned above provides a convenient criterion whose advantage is impersonalness. It is reasonable to suppose that this investigation gives a fair approximation to the needs of the college student. But it must be understood that the bibliography upon which this comparison is based cannot be considered exhausive, and that a check against an item is no indication of the depth of the treatment; for this investigation is not a weighing of this against that, but merely a survey to see what tendencies these authors are following.

It has seemed desirable to omit from this list such topics in algebra as are within the limit of the smallest amount assumed by any of these authors. This would include simultaneous linear equations, negative numbers, ratio and proportion, quadratics in one unknown, and simple formulas. Demonstrative geometry is assumed in all of them. In some cases, however, the review work includes these subjects and a list of important theorems in geometry.

In this table, the number after each item is the rank it would have in the report if the omissions noted above are made. An asterik (*) means that the subject is included in the text, a hyphen (-) means that it is omitted.[2]

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[1] Reorganization of Mathematics in Secondary Education Bureau of Education Bulletin, 1921, No. 32 pp. 37 and 38.

[2] The names of the texts corresponding to the numbers will be furnished to anyone who is interested in knowing them.

Topic	Rank in report	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
Imaginary numbers	23	–	*	*	*	–	–	*	*	*	–	*
Graphs												
Representation of statistical data	8	–	*	*	*	*	–	–	*	*	*	*
As a method of showing dependence	4	*	*	*	*	*	*	*	*	*	*	*
As a method of solving problems	12	–	*	*	*	–	*	–	*	*	*	*
The linear function $y = mx + b$	1	*	*	*	*	*	*	*	*	*	*	*
The quadratic function $y = ax^2 + bx + c$	6	*	*	*	*	*	*	*	*	–	*	*
Equations of higher degree	30	–	*	*	*	–	*	*	*	*	*	*
Variation	10	–	*	–	–	–	–	–	*	–	*	–
Numerical computation												
With approximate data	5	–	–	*	*	–	*	–	*	*	*	*
Short cut methods	20	–	–	*	*	–	*	–	*	–	–	*
Use of logarithms	3	–	*	*	*	*	*	–	*	*	*	*
Use of other tables	21	–	*	–	*	–	*	*	*	–	*	*
Slide rule	22	–	–	*	–	–	*	–	*	–	*	*
Exponents	13	–	*	*	*	*	*	–	–	*	–	*
Theory of logarithms	16	–	*	*	*	*	*	–	*	*	*	*
Arithmetic progression	24	–	*	–	*	–	–	–	*	–	–	*
Geometric progression	25	–	*	–	*	–	–	–	*	–	–	*
Binomial theorem	14	–	*	*	*	–	–	*	*	–	*	*
Probability	26	–	–	*	–	–	–	–	–	–	*	*
Statistics												
Meaning and use of elementary concepts	9	–	–	–	–	–	–	–	–	–	*	*
Frequency distributions	11	–	–	–	–	–	–	–	–	–	*	*
Correlation	17	–	–	–	–	–	–	–	–	–	*	–
Numerical Trigonometry	2	–	*	*	*	*	*	–	*	*	*	*
Trigonometry (usual course)	7	–	*	*	*	*	*	–	*	*	*	*
Analytics												
Fundamental concepts	18	*	*	*	*	*	*	*	*	*	*	*
Straight line	15	*	*	*	*	*	*	*	*	*	*	*
Circle	19	*	*	*	*	*	*	*	*	*	*	*
Conics	27	*	*	*	*	*	*	*	*	*	*	*
Polar Coordinates	28	–	*	*	*	*	*	*	*	–	–	*
Empirical curves	29	–	*	*	–	–	–	–	*	–	*	*

It should be noted that Books I and VII explicitly assume a knowledge of trigonometry and logarithms. Book VII presupposes work with approximate data, so also does II.

Differential calculus is treated in I, III, VI, VIII, X and XI; integral calculus in I, VI, VIII, X and XI.

A study of this table shows that all of these books either specifically assume or actually cover trigonometry and logarithms. Two omit work with approximate data, two assume it. All use the graph to show dependence, but only eight use the statistical graph.

It is surprising that the subject of statistics which ranks so high in the report, is treated in only two of the eleven texts. A probable explanation is that, devoid of the theory, the calculation of statistical measures is mere arithmetic; and the theory has the reputation of being exceedingly difficult. Book X has solved this problem in a very happy way. The work on the Theory of Measurement begins with a discussion of permutations, combinations, the binomial expansion as derived by the method of combinations, and, finally, probability. Thus the first part deals with events whose frequencies may be calculated by a consideration of the events themselves. The second part of the chapter contains work with data whose frequencies cannot be guessed by *a priori* reasoning, but where the information must be got by observing many cases. It is the distinction between calculating my chance of drawing the ace of spades from a complete pack of cards, and computing my chance of living to be seventy from a mortality table. The discussion of the variable quantities of the second type necessitates the use of frequency curves, measures of central tendency as medians, modes and the arithmetic mean, also quartile deviations and the mean square deviation. Finding the equation of a frequency curve representing a symmetrical distribution is a pretty piece of work with integral calculus. The notion of least squares is introduced as a prelude to finding regression lines, and the chapter ends with the calculation of the Pearson *r*. The whole piece of work necessitates the use of some trigonometry, some analytics and also differential and integral calculus. In this way, it makes an excellent "last chapter." It provides a splendid exposition of the subject for a person who craves some of the theory in small compass.

Book X is also conspicuous for the way in which problems using empirical data follow many of the theoretical discussions.

In Book XI, the chapter on Logarithms, and that on Progressions and Series, include some work in the mathematics

of finance. Compound interest, depreciation, the amount to be invested yearly at a given rate to yield a given amount at the end of a certain period, the present value of an annuity, etc., are certainly more interesting than the bouncing ball of our high school algebras, and if one is able to use logarithms, these problems are not difficult. They remind one of the typically French examples in Bourlet's *Algèbre,* where the thrifty paterfamilias is depicted as calculating how much he must invest each year to provide his year-old daughter with a suitable *dot* when she will be eighteen. It is significant that Book XI devotes nearly half as much space to these thrift problems as it gives to analytic geometry.

A particularly valuable part of Book IX is its discussion of the roots of a quadratic equation. This discussion is under four heads: problems admitting two solutions, problems with two solutions which are only apparently different, problems which admit of only one solution, problems which admit of no solution.

Book III is notable on several scores not the least of which is the fact that the notion of a derivative is introduced by easy stages from the very beginning of the book. At first it figures as the *change ratio* of a linear function. Later it comes in as the *slope* of the graph of a quadratic function, and it is used in rate problems and in finding maxima and minima. Finally, when the student becomes thoroughly acquainted with the idea of a derivative, he learns its name. Book III also has a valuable discussion of that vexing problem of approximate computations.

Two interesting points in textbook making are the use of timed exercises in Book IV, and a scheme of self-checking in VI where many of the groups of exercises are followed by the answers arranged in a different order.

On the score of arrangement, Book I and Book III are well toward the opposite ends of the scale. Book I begins and ends with material gathered from an Analytics textbook and a Calculus. The material has been somewhat recast, and the sections are bound together by work on *Curve Plotting* and *Functions and Graphs.* This book is perhaps the nearest of the eleven to being an abridged Analytics bound with an abridged Calculus. The arrangement of Book III on the other hand, shows how

completely the function idea dominates the work. First comes an introduction treating the notion of functions, then there is a review of algebra. In the book proper, there is a treatment of elementary functions which include linear, quadratic, cubic, trigonometric, logarithmic and exponential, and implicit quadratic functions. This is followed by the application of the subject to geometry in the straight line, circle, other conic sections, polar coordinates, parametric equations. Then come applications to algebraic methods, theory of equations, complex numbers, determinants, etc. Finally, for students whose previous training and ability warrant it, there is a discussion of the functions of two variables: i. e., Solid Analytic Geometry.

On the whole, then, some of these texts are close to the stratified course, some are closely unified. Some include surprisingly many of the topics listed as being of essential or of considerable importance in the investigation of the things useful in later college work; others have relatively few of them.

If one may make a guess, the future development of this work will be less and less along the conventional line. As one studies these text books, he almost invariably finds that the work in analytics is arid in comparison to the rest. One questions whether the normal probability curve though younger, may not in time usurp some of the attention paid to conics.

We may make private reservations as to whether the texts that appear to be unified will be so to the college freshman, and whether some of these authors have not crammed their year so full of new phases of mathematics that the student may be quite bewildered in mastering them. We may also wish we might ask certain of these teachers such questions as this: "What reason have you for teaching radian measure unless you plan to follow it directly with the differentiation of trigonometric functions?"

But for all one's doubts and cavilings, one point remains fixed. In contrast to our highly theoretical, arm-chair mathematics, the more radical of these books succeed in convincing us that freshman college mathematics may be made to present many contacts with the real world, and that, even though these contacts may be through problems that have only the appearance of reality, they at least have the virtue of showing the things that "mathematics has done and is doing for mankind."

TEACHING THE ALGEBRAIC LANGUAGE TO JUNIOR HIGH SCHOOL PUPILS[1]

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Progressive teachers of high school mathematics are today interested in two types of questions. First there are those questions having to do with the selection of material and with its organization into courses. In the second place, there are questions dealing with methods of presentation in text books and methods of teaching, the answer to which must be based upon an analysis of the subject matter to be taught and upon a study of the psychology of the mental processes involved. The paper read by Dr. Breslich here tonight is typical of the first type, the recent articles by Dr. Thorndike in The Mathematics Teacher are typical of the second. As the other paper on this evening's program dealt with the selection and organization of material, I thought it would be better for me to consider a question of the second type.

Several months ago there appeared in many of our newspapers a certain cartoon. The principal character was a small boy dressed in a neat black suit with stiffly starched collar and cuffs, wearing large horn rimmed spectacles, and carrying an armful of books. He had just moved to town and had been boasting to the other boys of his extraordinary scholarship. Among other things he had boasted that he could speak four languages—English, French, Spanish, and Algebra. The boys of the town endured his boasting as long as they could, but finally decided to put him to the test. So a self appointed committee waited upon the erudite stranger and asked him such questions, as "How do you say 'good morning' in Spanish?" and "How do you say 'pickles' in French?" The new boy came through with flying colors, as none of his inquisitors knew a word of any language except English, until one boy proposed the question, "How do you say 'bananas' in Algebra?" This was a "stumper" and the inquisition ended in a general melee with the new boy on the bottom of the pile.

No matter in what grade algebra is first taught, or whether it is presented as a separate subject or combined with arith-

[1] Read before the National Council of Teachers of Mathematics at Cleveland, Ohio, March 31, 1923.

metic, geometry and other portions of mathematics in some sort of a general or correlated course, in any case the text book writer and the teacher are confronted with the problem of helping the pupils master a new notation, a new language. Figuratively speaking, the pupils must be taught to say "bananas" in algebra. When algebra is taught in the junior high school this phase of the work assumes even greater importance.

The learning of the algebraic language is, without doubt, the greatest difficulty presented to the beginner by the subject. The success of our text book writers and of our teachers in helping students to master this new language has not been very great, to say the least, as witnessed by the high rate of infant mortality in the subject. Too many pupils "go through" a course in high school algebra without ever grasping the meaning of the subject and certainly without gaining any real power. Manipulation and the blind following of mechanical rules are too often substituted for understanding and mastery. The pupils speak very glibly of "cancelling," "transposing," "clearing of fractions," "cross multiplying," etc., etc., but do not understand what any of these terms really mean. Many habits must be formed in algebra, it is true, but these habits must be built on a real understanding of the purpose of the subject, of its important methods of procedure, and of the fundamental principles on which these methods are based, if they are to be of any real use to the pupil in gaining a true mastery of the subject. To do otherwise is to build our mathematical foundation on sand and the structure is sure to fall of its own weight sooner or later.

Our lack of success in initiating pupils into the meaning of algebra has been partly, if not largely, due to a lack of appreciation of the difficulties presented to the beginner by the notation or language of the subject, and to a lack of clear understanding of the nature of this language and particularly of the psychological difficulties that it presents to the learner. It is the purpose of this paper to attempt to break up "the mastery of the algebraic language" into its component abilities and to make some suggestions as to methods of developing these.

The mastery of a new language, such as French for example, presents two types of difficulties (1) the acquisition of a new vocabulary and (2) the learning of the grammar and the sentence structure so that one can make connected state-

ments. Many of our soldiers in France acquired quite an extensive French vocabulary but made little progress in the direction of connected expression of their thoughts. The algebraic language presents both types of difficulties. The pupil must become acquainted with the vocabulary of the subject, with the use of letters and mathematical symbols to represent various related quantities, and must also study the sentence of algebra, the expression of relationships between quantities by means of algebraic equations.

In studying a new language it is easier to translate from the new into the old, than the other way around, and such work usually comes first, but practice of both kinds is necessary to a complete mastery. So in algebra it is easier to translate from algebra into English than from English into algebra, and such practice should be given first as a preparation for the more difficult work to follow.

I have attempted, by drawing this rough parallel between the learning of the language of algebra and the learning of a new language such as French, to show that the complete mastery of the algebraic language involves four separate abilities. These, stated in the probable order of their difficulty, are:

1. The ability to translate algebraic representations of quantities into English.
2. The ability to translate English descriptions of related quantities into algebraic expressions.
3. The ability to translate algebraic statements of relationship (equations and formulas) into English sentences.
4. The ability to translate English statements of relationships into algebraic equations and formulas.

I want next to consider, briefly, the type of practice that must be provided in order to develop each of these abilities. In order to understand and use intelligently a formula such as $(a+b)^2 = a^2 + 2ab + b^2$ the pupils must know that $(a+b)^2$ is the algebraic shorthand for "The square of the sum of any two quantities." To understand and use the formula $A = \frac{1}{2}a(b+b')$ they must know that $(b+b')$ is the algebraic equivalent of "The sum of two bases" and that $\frac{1}{2}a(b+b')$ is algebra for "Half the product of the altitude and the sum of two bases." Plenty of practice should be given the pupils on translating the alge-

braic representations of quantities into English. The character of this practice is indicated by the following short list of drill examples.

Drill Exercises to Give Practice on Translating Related Algebraic Expressions into English

(Type 1a)

1. If d represents the number of dollars paid for a sled, what does $2d$ represent, $5d$?

2. If n represents one number what does $n+6$ represent? $n-3$? $\frac{1}{2}n$? $\frac{2}{3}n$?

3. If x represents A's age in years today what does $x-3$ represent? $x+5$?

4. If y represents A's age ten years ago what does $y+6$ represent? $y+10$? $y+12$?

5. If n represents a certain number what does $n+3$ represent? $n(n+3)$? $(n+3)-n$? $\dfrac{n}{n+3}$? $\dfrac{n+3}{n}$?

6. If a and b represent any two numbers what does $a-b$ represent? $(a-b)^2$?

For convenience drill material of the above type will be referred to hereafter as Type 1a.

Somewhat more difficult than the preceding is the work of representing related quantities by means of algebraic expressions. The ability to do this is fundamental to the solution of verbal or written problems. If the problem is capable of solution enough facts about these quantities and the relations existing between them must be known or given to enable the pupil to express each of the unknowns algebraically. Consider, for example, the problem "John bought a tablet and a pencil. The tablet cost 5c more than the pencil and both together cost 13c. Find what he paid for each." This problem involves two unknowns, the number of cents paid for the pencil and the number of cents paid for the tablet. The first difficulty encountered by the pupil is that of building up the algebraic vocabulary of the problem. If he lets c represent the number of cents paid for the pencil, he must be able to see that $c+5$ may be used to represent the number of cents paid for the tablet, since the

tablet is known to have cost 5c more than the pencil. Or, if he prefers, he can let $13 - c$ represent the number of cents paid for the tablet as it is known that the tablet and pencil together cost 13c. Drill material such as that given in the following short list should be used consistently throughout the course to develop the ability to represent quantities by means of letters and algebraic symbols.

Drill Exercises on Translating English Descriptions of Related Quantities into Algebraic Expressions

(Type 1b)

1. If n represents a certain number what will represent a number 5 times as large? Half a large? Three-fourths as large? Seven less? Nine more?

2. If y represents the number of years in John's age what will represent the number of years in his father's age if John's father is just 30 years older than John? What will represent the age of John's older brother if he is 5 years older than John? What will represent the age of his sister who is 2 years younger? Of his grandfather who is just twice as old as John's father? Of his grandmother who is 5 years younger than his grandfather?

3. If C is now y years old how old was he 7 years ago? How old will he be in 3 years?

4. If a train runs at the rate of 28 miles an hour, how far does it run in 3 hours? In h hours?

5. If x represents the number of inches in the side of a square what represents the number of square inches in the area?

6. A rectangle is 6 feet longer than it is wide. If w represents its width what represents its length? Its perimeter? Its area?

7. If a and b represent any two numbers what will represent their sum? The square of their sum?

This type of drill material will hereafter be referred to as Type 1b.

The ability to grasp and express relationships is one of the most important in all of mathematics. In connection with their

high school mathematics, pupils should be trained in four methods of expressing relationships.

1. In words.
2. By means of statistical tables.
3. By means of graphs.
4. By means of algebraic equations and formulas.

The entire course in secondary mathematics should be built around the study of relationships and methods of expressing them. This, to me, is what is meant by the National Committee and others when they suggest that the "function concept" be made the unifying element of all work in mathematics. I have, so far, purposely avoided using the term "function concept" as I find that to many it is meaningless and to others it means something highly abstract and formal that would be entirely out of place in high school mathematics.

Of the four methods of expressing relationships mentioned above only the fourth comes properly within the scope of this paper. The equation $2x + 7 = 15$ should say to the pupil "Two times a certain number is 7 less than 15, what is the number?" the formula $(a + b)^2 = a^2 + 2ab + b^2$ should say "The square of the sum of any two quantities is always equal to the square of the first quantity, plus twice the product of the two, plus the square of the second." In order that these and similar equations and formulas may have such meanings to the pupils they must be given practice in translating formulas and equations into English statements of relationship. To many high school students, yes, to many college freshmen, the formula $a^2 - b^2 = (a + b)(a - b)$ does not state a general relationship and does not give a general set of directions for factoring a certain type of expressions. Instead of saying, as it should, "The difference of two squares always factors into the square root of the first plus the square root of the second, times the square root of the first minus the square root of the second" it simply says "a^2 minus b^2 equals the quantity a plus b times the quantity a minus b." The type of practice material needed to develop this ability to interpret equations as statements of relationships existing between the quantities involve is shown by the short list that follows. This type of material will be referred to hereafter as Type 2a.

Drill Exercises on Translating Algebraic Equations and Formulas into English

(Type 2a)

1. Translate the following equations into English if in each n represents a certain number.

a. $n + 3 = 7$
b. $n - 5 = 9$
c. $2n = 18$
d. $\frac{1}{2}n = 7$
e. $\frac{2}{3}n = 5$
f. $2n - 1 = 9$
g. $3n + 7 = 19$
h. $7 - n = 2$

2. State the following equations in English if y represents the number of years in John's age and $y + 30$ represents the number of years in his father's age:

a. $y + (y + 30) = 60$
b. $(y + 30) = 3y$
c. $y = \frac{1}{3}(y + 30)$
d. $(y + 30) + 2y = 75$
e. $2(y + 30) - 3y = 45$

3. If L represents the number of feet in the length of a rectangle and $L - 7$ the number of feet in the width, translate the following equations into English.

a. $L(L - 7) = 18$
b. $2L + 2(L - 7) = 22$

4. Translate the following formulas into words:

a. $V = \dfrac{\pi d^3}{6}$, if V represents the volume, and d the diameter of a sphere.

b. $L = \frac{1}{2}S(C + C')$, if L represents the lateral surface, S the slant height, and C and C' the circumferences of the two bases of the frustrum of a right circular cone.

c. $A = P + I$, if A represents the amount, P the principal and I the interest.

d. $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$

e. $(x - y)^2 = x^2 - 2xy - y^2$

The ability to express relationships by means of algebraic formulas or equations is essential to success in the practical applications of mathematics. In order to use algebra as a tool to aid one in meeting a practical problem one must be able to size up the situation, to represent the quantities involved by

algebraic expressions, and finally to state the relationships between these quantities in the form of an equation, or equations, as the case may be. This ability is fundamental to the solution of the written or verbal problem. Indeed the chief value of such problems is to be found in the practice they afford in algebraic representation of quantities and algebraic statement of relations. In solving a verbal problem, after the pupil has built up his algebraic vocabulary, has represented the unknown quantities in the problem by means of algebraic expressions, the next step is to state the remaining known relationship between these quantities in the form of an algebraic equation. In the problem previously used as an illustration, the pupil has, we will say, let C represent the number of cents paid for the pencil and $C + 5$ the number of cents paid for the tablet. The fact that they both together cost 13c has not yet been used and must be expressed as an algebraic equation, as $C + (C + 5) = 13$. When the pupil discovers that "The area of a triangle is equal to one-half the product of the altitude and the base," he should be able to translate this rule into the formula $A = \frac{1}{2}ab$. This ability can be developed by practice such as that suggested in the short list below. This type of drill material will be referred to hereafter as Type 2b.

Drill Exercises on Translating English Statements of Relationship into Algebraic Equations and Formulas

(Type 2b)

1. Express the following statements as equations:

a. The sum of x and 3 is 5.

b. If 7 is subtracted from n the result is 4.

c. x is 7 less than 12. (Two ways.)

d. y is 3 more than 8. (Two ways.)

e. 32 is 4 times x.

f. If n is subtracted from 52 the result is the same as that obtained by adding 18 to n.

g. Three times n is 7 more than 13. (Two ways.)

2. If d represents the number of dollars A has and $d + 7$ represents the number of dollars B has, express the following statements as equations:

a. A and B together have $17.

b. A has $5/12$ as much as B.

c. If B gives A twice as much as A has to start with B will have \$2 left.

d. If B gives A \$3.50 they will both have the same amount.

3. State each of the following in the form of a formula:

a. The square of a fraction equals the square of the numerator divided by the square of the denominator.

b. To divide a fraction by an integer multiply the denominator by the integer.

c. The distance covered in a given time by a body moving at a uniform rate is equal to the product of the rate and the time.

d. The reading on a Fahrenheit thermometer is 32° more than $9/5$ of the reading on a Centigrade thermometer.

e. The arithmetical average of two numbers is equal to the sum of the numbers divided by 2.

f. The product of the sum and difference of two quantities is always equal to the square of the first quantity minus the square of the second quantity.

Ten or fifteen years ago our text books in beginning algebra did little towards helping the pupils master the algebraic language. Since that time a partial reform has come and most texts now devote more or less time to practice intended to develop this mastery. There is, however, still room for much improvement in this respect. To show this, I have made an analysis of seven different texts that include a presentation of beginning algebra. Three of these are typical algebras written for use in the ninth school year, one is a text on general or correlated mathematics intended for the same year, and the other three are junior high school texts for use in the seventh, eighth and ninth years. The results of this analysis are given, for the several books, in the following tables which show the number of drill examples given and classify them into the four types previously described.

Table Showing the Number and Type of Examples Giving Practice on the Algebraic Language in Book A (Algebra)

	Algebraic Representation of Related Quantities			The Expression of Relations by Means of Algebraic Equations		
PAGE	Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
1-25	3	32	35	2	2	4
26-50	0	0	0	0	0	0
51-75	0	34	34	1	1	2
76-100	0	34	34	0	1	1
101-125	0	21	21	0	2	2
126-150	0	8	8	0	2	2
151-175	0	0	0	0	0	0
176-200	0	14	14	0	3	3
201-	0	9	9	0	17	17
Total	3	152	155	3	28	31

Book B (Algebra)

	Algebraic Representation of Related Quantities			The Expression of Relations by Means of Algebraic Equations		
PAGE	Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
1-25	0	37	37	0	0	0
26-50	3	17	20	0	14	14
51-75	0	34	34	0	15	15
76-100	0	28	28	0	17	17
101-125	0	0	0	0	0	0
126-150	0	0	0	0	9	9
151-175	0	0	0	0	0	0
176-200	0	10	10	0	9	9
201-	1	13	14	0	4	4
Total	4	139	143	0	68	68

Book C (Algebra)

	Algebraic Representation of Related Quantities			The Expression of Relations by Means of Algebraic Equations		
PAGE	Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
1-25	0	36	36	2	0	2
26-50	0	14	14	2	0	2
51-75	0	20	20	0	2	2
76-100	1	4	5	0	1	1
101-125	0	1	1	0	0	0
126-150	0	0	0	0	0	0
151-175	0	5	5	0	0	0
176-	0	15	15	0	0	0
Total	1	95	96	4	3	7

Book D (General Mathematics)

PAGE	Algebraic Representation of Related Quantities: Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	The Expression of Relations by Means of Algebraic Equations: Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
1-25	0	0	0	0	0	0
26-50	0	34	34	2	4	6
51-75	0	1	1	0	0	0
76-100	0	2	2	4	3	7
101-125	0	19	19	0	4	4
126-275	0	5	5	2	0	2
276-300	0	0	0	3	28	31
301-	0	0	0	0	8	8
Total	0	61	61	11	47	58

Book E (Junior High School Series)

BOOK AND PAGE	Algebraic Representation of Related Quantities: Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	The Expression of Relations by Means of Algebraic Equations: Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
Bk. I						
1-25	0	11	11	0	24	24
26-50	0	5	5	0	8	8
51-75	0	0	0	0	1	1
76-100	0	0	0	0	9	9
101-125	0	0	0	0	1	1
126-150	0	5	5	0	0	0
151-175	0	4	4	0	3	3
176-	0	0	0	3	3	6
Bk. II.						
1-25	0	38	38	23	38	61
26-50	1	6	7	5	51	56
51-75	0	0	0	0	8	8
Bk. III.						
1-25	1	27	28	1	22	23
26-50	0	3	3	0	0	0
51-75	0	4	4	1	3	4
76-100	0	5	5	0	0	0
101-125	0	0	0	0	0	0
126-	0	2	2	0	11	11
Total	2	110	112	33	182	215

Book F (Junior High School Series)

BOOK AND PAGE	Algebraic Representation of Related Quantities: Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	The Expression of Relations by Means of Algebraic Equations: Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
Bk. I.	0	0	0	0	0	0
Bk. II.						
1-25	11	22	33	12	28	40
26-50	1	0	1	7	10	17
51-75	0	0	0	0	9	9
76-100	0	2	2	0	0	0
101-	0	8	8	0	12	12
Bk. III.						
1-25	14	0	14	8	23	31
Total	26	32	58	27	82	109

Book G (Junior High School Series)

BOOK AND PAGE	Algebraic Representation of Related Quantities: Algebra into English (Type 1a)	English into Algebra (Type 1b)	Total	The Expression of Relations by Means of Algebraic Equations: Algebra into English (Type 2a)	English into Algebra (Type 2b)	Total
Bk. I.						
1-175	0	0	0	0	0	0
176-200	0	0	0	6	0	6
Bk. II.						
1-25	0	0	0	9	0	9
26-50	0	0	0	1	0	1
51-75	0	0	0	0	0	0
76-100	0	0	0	1	4	5
101-	0	0	0	12	0	13
Bk. III.						
1-25	0	32	32	0	30	30
26-50	0	18	18	0	0	0
51-	0	0	0	1	8	9
Total	0	50	50	30	42	72

Summary

Book	Type of Book	School Years	1a No.	1a %	1b No.	1b %	1a and 1b No.	1a and 1b %	2a No.	2a %	2b No.	2b %	2a and 2b No.	2a and 2b %	Total
A	Algebra	9th	3	1.6	152	81.6	155	83.4	3	1.6	28	15.1	31	16.7	186
B	Algebra	9th	4	1.9	139	65.9	143	67.8	0	0.0	68	32.2	68	32.2	211
C	Algebra	9th	1	1.0	95	92.2	96	93.2	4	3.9	3	2.9	7	6.8	103
D	Gen.Math.	9th	0	0.0	61	51.4	61	51.4	11	9.2	47	39.5	58	48.7	119
E	Jr.H.S.	7,8,9	2	0.6	110	33.6	112	34.2	33	10.1	182	55.6	215	65.7	327
F	Jr.H.S.	7,8,9	26	15.6	32	19.2	58	34.8	27	16.2	82	49.1	109	65.3	167
G	Jr.H.S.	7,8,9	0	0.0	50	41.0	50	41.0	30	24.6	42	34.4	72	59.0	122

The total number of drill examples on the use of the algebraic language varies from only 103 in Book C (an algebra) to 327 in Book E (a junior high school series). The number of examples of Type 1a varies from none in Book D (general mathematics) and Book G (junior high school series), to 26 in Book F (junior high school series); and from 0.0% of the total number of drill examples on the algebraic language in the case of Books D and G, to 15.6% of the total in the case of Book F.

The number of examples of Type 1b varies from only 32 in Book F (junior high school series) to 152 in Book A (algebra); and from 19.2% of the total in Book F (junior high school series) to 92.2% of the total in Book C (algebra).

The number of examples of Type 2a varies from none in Book B (algebra) to 33 in Book E (junior high school series); and from 0.0% of the total in Book B to 24.6% of the total in Book G (junior high school series). The number of examples of Type 2b varies from 3, or 2.9% of the total, in the case of Book C (algebra) to 182, or 55.6% of the total in Book E (junior high school series).

It is evident from this analysis that some of our texts do not give sufficient practice in the algebraic language and that there is no agreement as to the relative importance of the four types of drill. The general mathematics text and the junior high school books show one decided improvement over the algebras, namely, increased emphasis on the expression of relationships (Types 2a and 2b). These books show an average of 113.5 examples of this type as against 35.3 in the algebras. There is still room, however, for further improvement in the matter of providing systematic drill on the algebraic language.

THE UNITARY ORGANIZATION OF THE MATHEMATICS OF THE SEVENTH, EIGHTH, AND NINTH GRADES[1]

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Logical Units of Instruction. In the traditional courses in mathematics there has been a constant tendency in organization of materials to organize in *logical* units, *i. e.*, to put together subject-matter which *logically* belongs together. In studying the material, the pupil concentrates intensely on one topic at a time without giving thought to its relation to other topics or to the course as a whole. Moreover, he does not concentrate fully enough on a single topic to get the experiences necessary to attain complete mastery. For, the *unit* is simply a lesson on a fact, a process, or a principle, taught and studied on a certain day, without clear connection with other related facts, and without definite insight as to what each fact contributes to the whole. The goal of instruction is ability of the pupil to recite on, or to discuss, the particular lesson taught, before taking up the study of the next lesson. As a result, the pupil may answer satisfactorily questions on each lesson, without being able to recite on the topic as a whole, and without being able to distinguish between the relative values of the parts of a topic. A successful recitation is considered sufficient evidence that a lesson is learned. Tests consist of questions on isolated facts, rather than on the unit as a whole.

Examples of this type of organization are not hard to find. In college algebra we find students studying in succession synthetic division, depression of equations, diminishing and multiplying the roots of equations, relations between the roots and the coefficients, graphic representation, and many other isolated processes and principles, without conceiving their relations to the topic, the finding of the roots—integral, fractional, irrational, or complex—of rational integral equations of the nth degree in one unknown. Understanding of these relations and of the aims and purposes of the various processes, if attained at all, comes at the end, rather than at the beginning of the unit. Similarly, in geometry we find in one and the same chapter

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[1] Read before the National Council of The Teachers of Mathematics, February 28, 1923, Cleveland, Ohio.

separate and fragmentary treatments of such topics as triangles, parallel lines, loci, quadrilaterals, and others. The student studies a mass of unrelated facts, theorems, and problems, each of which is to be learned, mastered, and tested separately.

This type of organization is not satisfactory in secondary school work. Adults are frequently heard to remark that they recall less of the mathematics which they have studied in the high school than of any of the other subjects, although they always considered themselves good students in mathematics. The explanation of this may be found in the fact that many high school students learn their lessons in mathematics daily and are able, on the next day, to give a satisfactory report on each lesson without having grasped the few really important principles contained in courses in algebra or geometry. In fact, they are mastering only assimilative material, or applications of these few principles, and not the broad principles, with the result that such material is not permanently retained.

Pedagogical units. To attain lasting results the organization must be made in *pedagogical* units, rather than in logical units, *i. e.*, material must be put together which is most economically and effectively learned together. We are just learning how to do this in mathematics. Under the leadership of Professor H. C. Morrison considerable work along this line has been done in several departments in the University High School. For example, the subject-matter of a course in elementary physical science has been divided into units, each of which is organized around a naturally related group of phenomena, such as weather, fire, fuel, building material, heating, etc. Since the concept of weather is a complex of air pressure, winds, humidity, evaporation, rain, snow, temperature, clouds, etc., this concept may therefore be organized as a teaching unit "weather." The goal of instruction under this plan is not only the ability of the pupil to discuss some abstract principles, such as the laws of evaporation, or electro-magnetism, but rather a clear picture in his mind of the major factors of weather and their relationships, or of the construction and operation of the different instruments used to communicate messages by electric current.[1]

[1] Beauchamp, Wilbur L. A preliminary experimental study of technique in the mastery of subject-matter in elementary physical science. Studies in Secondary Education, Supplementary Educational Monograph No. 24, The University of Chicago, January, 1923, pp. 54, 55.

Likewise, in history a year's course in the *Survey of Civilization* has been organized in such large units as primitive life, oriental civilization, Greece, Rome, the middle ages, the crusading movement, beginnings of the modern world, colonial expansion, and the new world. "This arrangement gives the pupil a clear conception of the great movements in history and an adequate understanding of typical civilization of the past."[1]

By concentrating attention on the great movements and large features of history the study of what is typical in the civilization of the past gives emphasis to the elements which are of permanent value in history.

Some of the authors of text books in secondary school mathematics are beginning to arrange subject-matter in such pedagogical units. A very good example of this type of organization in geometry is the topic which is concerned with the measurement of angles in terms of arcs of a circle. Before beginning to study the detailed body of related material, the teacher gives the pupil a view of the topic as a whole. Thus, he explains the meaning, purpose, and value of this method of measuring. He shows the various positions of the sides of the angle with reference to the circle. By means of drawings, he illustrates that the vertex of the angle may be within the circle, on the circle, or outside of the circle. In the first case, it may fall on the center or on some other point. In the second case, both sides may be chords, or one may be a chord and the other a tangent. In the third case, both sides may be tangents or both secants, or one may be a tangent and the other a secant. As the drawings are made to represent the various cases possible, the theorems corresponding to the cases are formulated. This whole presentation can be given in less than twenty minutes. It is seen that instead of a number of isolated theorems, the pupil learns a series of theorems which he recognizes as related to each other, and to the topic as a whole. Even now he feels that he has a very good conception of the unit, and he is able to write out the story presented by the teacher. Furthermore he sees that the methods of proof of the various theorems are likely to be the same, except for slight modifications. He will therefore master all of the proofs with economy of effort, and he will retain

[1] Hill, Howard Copeland and Barnard, Arthur Fairchild. Curriculum in History. Supplementary Educational Monographs No. 24, The University of Chicago, January, 1923, p. 95.

them more permanently than is possible when each theorem is studied separately. Returning to the example above in college algebra, the teacher selects as the unit the solution of equations, and then begins the study of the unit by presenting to the class the general problem of solving equations of a certain type. He calls attention to the nature of the roots, and to the fact that a variety of methods of procedure must be learned which are to be employed according as the roots may be integral, fractional, rational, irrational, or complex. When the student later studies these methods, he will at no time lose sight of the major topic, he will know exactly how the various facts are related to each other and how they apply to the general problem. This will not only increase his interest, and make his study purposeful, but will help to make retention permanent.

The problem of the junior high school is largely a problem of curriculum construction. There has been a great deal of discussion on the content of the courses in mathematics, and it is now generally agreed that a certain body of material taken from the fields of arithmetic, algebra, geometry, and trigonometry be included. However, the problem of arranging this material offers great difficulty. It is quite possible that the solution may be found in the organization in pedagogical units as the most suitable for instruction of the pupils of junior high school age. It will not be attempted, in one paper, to submit a detailed organization for the whole three-year course, nor even for a year's course. However, at least one unit will be submitted in detail for consideration.

Test of a Pedagogical Unit. The following may be applied as a test of a pedagogical unit.

1. It must be possible to present the unit as a whole in concise form, giving the learner a clear, general conception of the unit.

2. It must be a compact body of closely related facts and principles susceptible of mastery and of being tested for mastery.

The first test implies that at the beginning each unit should be so presented in a concise expository survey that the class sees clearly the road it is going to travel, getting a complete, even though only superficial, notion of the unit itself. The effect of the presentation will contribute much toward making thinking purposeful and learning permanent. This presentation is

to be followed by assimilation, where the class absorbs and assimilates the principles taught, and where each pupil makes them his own through the wealth of experiences and of applications provided. Special care must be taken that a unit in the teaching process is not being taught as a group of separate sub-units.

The second test above implies that at the end of each unit the class has not only mastered each of the various principles taught, but that each member shall be able to give a substantial talk, or to write a respectable paper, on the unit as a whole.

Eight to ten such units ordinarily should form a year's work.

Plan of Organization. In view of the fact that geometry is the most concrete material, and also the most available material, intuitive geometry should be made the means of unifying a year's work in the seventh or eighth grade. Furthermore, the physical and mental development of the child resembles that of the race, and therefore in teaching and learning we should follow the progress of the race, cutting down long periods of time to months, or days, or even minutes. Accordingly, the fact that geometry was developed before algebra, indicates another valid reason for teaching it first. The general aim of this geometry should be training in space relationships. Actual measurement of the things about us should furnish the experiences for the pupil. Through these it is easy to show to the pupil the necessity of making a study of the basic solids: the cube, rectangular block, prism, pyramid, cone, cylinder, and sphere.

Outline of a Year's Work. During the first class period the teacher, making use of wooden or cardboard models, briefly presents the aims of the year's work to the class. He makes clear why the pupils need to know all about these solids. He shows that the study necessitates a preliminary study of the perimeters and areas of certain plane figures, such as the triangle, square, rectangle, parallelogram, trapezoid, and circle, and that this, in turn, calls for an understanding and clear conception of angles and line-segments. Accordingly, the first units of instruction which naturally present themselves are: line-segments, angles, triangles, quadrilaterals, circles, areas, and volumes. From the beginning, the pupil is led to see, and continues to see, the relation of each unit to the general problem of the year's work.

Method of Instruction. To illustrate the method of instruction the unit *circle*, may be selected for particular detailed consideration. Here, as in the teaching of any unit, the first step is to determine the amount of information the class possesses as to the unit to be studied. This knowledge, which is necessarily incomplete and superficial, is obtained by the teacher through a series of questions. The following illustrate the type of questions to be asked: How is a circle drawn? What uses are made of a circle? What is a circle? How may we find the length of a circle? How can the content of the interior be measured? These and many similar questions bring out the fact that some pupils know a great deal about the circle, while others have but a very vague conception of its properties. Furthermore, through these questions the interest of all of the pupils is aroused, especially if it is shown that in general they do not have but need to have exact notions about the subject. It is well worth while to spend about half of a class period, or over, on this phase of instruction.

The teacher now presents to the class, in the time of fifteen or twenty minutes, and in as clear and direct a manner as possible, the main points to be studied in the unit. In the case of the circle he should show clearly that the circle is a closed curved line, every point of which has a fixed distance from the center. From a variety of designs the pupil learns about the use of the circle for ornamental purposes. It is explained to him that places on the surface of the earth are determined by means of imaginary circles. He learns that gas and electric meters enable the consumer to measure these utilities by means of circles. Circular graphs are shown to represent arithmetical facts in a clear and simple manner. He sees that careful geometric drawings may be made by means of circles. Through a few interesting problems the teacher shows the need for a formula with which to determine the length or the area of the circle. At the end of this presentation the pupil must be able to write a paper covering the teacher's talk, and showing that a clear and counit, the teacher goes over the topic a second time with them, herent conception of the unit has been obtained. If this test shows that some of the pupils lack a clear understanding of the varying the presentation according to the needs or difficulties of the pupils concerned. Sometimes it may be necessary to

present the unit a second time to the whole class before complete mastery is attained. The aim is to prepare the way, to enable every pupil to study the details in the unit intelligently and effectively.

The pupil then begins his study of details under the direction of the teacher. This is the time for supervised study. The pupil obtains, from the reading of the text book, an exact knowledge of all the essential facts, processes, or principles. During this time the teacher assists individuals in their difficulties, gives general directions to the class if they are needed by the class as a whole, and develops the study habit of the pupils. Occasionally the work of the class is stopped to listen to oral reports given by one or more pupils. This is not done to test these pupils, but for the purpose of class discussion. Such reports should be followed by questions and criticisms, or by a cross-examination by the teacher. The class then again returns to the study of the unit. Free use is made of all the mathematical subjects arithmetic, algebra, geometry, or trigonometry, wherever and whenever they can be made to contribute to clearness and understanding.

At the end of this study (without a formal review) a written test of the minimum essentials is given, in which every member of the class must write a satisfactory paper, showing that he thoroughly understands the topic. If a pupil fails in this test, he must study the topic further, after which he takes another test comparable with the first. If he still fails, the process is repeated, until he succeeds.

Minimum Essentials. The minimum essentials in this unit are the following abilities:

1. To use the compass to make designs containing circles or circle arcs.

2. To locate places on a map by means of latitude and longitude.

3. To read gas or electric light meters and to check the correctness of the amount of a gas bill.

4. To interpret circular graphs.

5. To make the fundamental constructions of plane geometry.

6. To apply the formulas $c = \pi d$ and $a = \pi r^2$ in the solution of problems in which either c or d are unknown.

7. To estimate the degree of accuracy to be attained in such problems.

8. To master the processes of abbreviated multiplication and division.

9. To solve word problems involving the notions of circumference and area of a circle.

10. To represent graphically the equation $c = \pi d$.

It should be understood that the actual experiences of the pupil cover a much wider field than is indicated by these essentials, but that he is not held for this material in a test. Thus, the pupil learns to compute local time from a given longitude. He not only reads gas meters, but makes drawings of gas meters corresponding to given readings. He makes circular graphs to represent tables of arithmetical data. He applies the fundamental constructions to a variety of problems of construction, etc., but all of this material is used mainly to supply wide experiences with those facts which are of fundamental importance, and mastery of which is to be attained.

Summary. In conclusion, the values of organizing in pedagogical units may be summarized as follows:

1. It is an effective method of bringing home to the pupil the importance, meaning and appreciation of a large topic as a whole.

2. It brings about appreciation of the relationships of facts and principles to each other and to the topic as a whole.

3. It supplies a motive for studying these facts and thus arouses the interest of every pupil.

4. Pupils are aware of the aims and purpose of the topics which they study, and secure mastery of each topic.

5. Complete failure becomes unknown as every pupil attains mastery of each topic studied.

6. During supervised study the teacher has unusual opportunity to develop the study habits of the pupil.

7. Much time is saved by avoiding unnecessary reviews, because the pupil is ready for the final test as soon as he finishes the study of the unit.

8. The pupil's power to retain what he has learned is greatly increased.

MECHANICS

Rectilinear Motion of a Point. Gravity.

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Distance. In the last article we were concerned with finding the final velocity, being given the initial velocity, acceleration and the time. We did not consider the distance passed over during any interval of time.

Let us take the problem where the initial velocity is 40 ft./sec. and the final velocity is 60 ft./sec. and the time given for this increase in velocity is 5 seconds. We can find the distance passed over as follows: First, find the acceleration by the final velocity formula.

$$v = v_1 + at \quad \left| \begin{array}{l} v = 60 \text{ ft./sec.} \\ v_1 = 40 \text{ ft./sec.} \\ t = 5 \text{ sec.} \end{array} \right.$$

$$\begin{aligned} 60 &= 40 + 5a \\ 5a &= 20 \\ a &= 4 \text{ ft./sec.}^2 \end{aligned}$$

Second, find the final velocity at the end of each second, and the distance passed over during each second. We will put it in tabular form.

Velocity at the start	= 40 ft./sec.
Velocity at the end of the first second	= 44 ft./sec.
Velocity at the end of the second second	= 48 ft./sec.
Velocity at the end of the third second	= 52 ft./sec.
Velocity at the end of the fourth second	= 56 ft./sec.
Velocity at the end of the fifth second	= 60 ft./sec.

How many feet did the body go the first second? It went more than 40 feet and less than 44 feet. Since the increase in velocity was uniform during the second, the distance passed over must have been 42 feet which is the average velocity during the second. In like manner we get the distances passed over during the second second, third second, fourth second, and the fifth second as 46 feet, 50 feet, 54 feet, and 58 feet respectively. If we add these distances we get 250 feet. This method is too long. Let us see if we can use the uniform velocity formula, $s = vt$.

If we find the average velocity, we will have the uniform velocity that the body could have gone at during this interval

of time and covered the same distance as though it had uniform accelerated motion.

We have given a method of finding the average velocity in the first paper:

$$\text{average velocity} = \frac{v_1 + v_2 + v_3 + v_4 + v_5 \ldots v_t}{t}$$

$$\text{or average velocity} = \frac{v_1 + v_2 + v_3 + v_4 + v_5 \ldots v_n}{n}$$

This latter formula might be better, for in the first one we might think we divided the sum of the velocities by the time.

In our case the average velocity is gotten as follows:

$$\text{average velocity} = \frac{40 + 44 + 48 + 52 + 56 + 60}{6} \; 50 \text{ ft./sec.}$$

This is the uniform velocity that the body could have gone at and covered the same distance. We will now use the uniform velocity formula.

$$s = vt \quad \left| \begin{array}{l} v = 50 \\ t = 5 \\ s = ? \end{array} \right.$$

$$s = 50 \cdot 5 = 250 \text{ feet. Ans.}$$

If v_1 = initial velocity, $v = v_1 + at$ = final velocity, a = acceleration, t = time and s = distance. The velocity at the start and the velocities at the end of each second form an arithmetical progression. The sum for an arithmetical progression is given by the formula, $s = n/2(a + 1)$. $a = v_1$ = first term, $1 = v_1 + at$ = last term and $n = t$ = the number of terms.

The average of n terms of an arithmetical progression is the sum divided by n.

Average $n/2(a + 1)$ divided by n or $\frac{1}{2}(a + 1)$.

But since we have an arithmetical progression all we have to do is to divide the sum of the first and last term by 2.

$$\text{average velocity } \frac{v_1 + v_1 + at}{2} \text{ or } v_1 + \tfrac{1}{2}at.$$

Now if we place $v_1 + \frac{1}{2}at$ for v in the formula for uniform motion, we have $s = (v_1 + \frac{1}{2}at)t$ or $s = v_1t + \frac{1}{2}at^2$.

If we square $v = v_1 + at$ we get $v^2 = v_1^2 + 2v_1at + a^2t^2$. Now we can write the last equation as follows: $v^2 = v_1^2 + 2a(v_1t + \frac{1}{2}at^2)$. But $v_1t + \frac{1}{2}at^2$ is equal to s from the other formula.

Therefore, $v^2 = v_1^2 + 2as$.

Now our three formulas for uniform accelerated motion are:

$$v = v + at.$$
$$s = v_1t + \frac{1}{2}at^2.$$
$$v^2 = v_1^2 + 2as.$$

If the initial velocity is equal to zero we have the following formulas:

$$v = at.$$
$$s = \frac{1}{2}at^2.$$
$$v^2 = 2as.$$

Let us now work out a problem by means of these formulas.

A stone is projected vertically upward with a velocity of 140 feet per second, and two seconds later another is projected on the same path with an upward velocity of 135 feet per second. When and where will they meet?

They will meet at some distance D from the ground. Each stone will reach this point twice provided they do not meet. They will reach this point on the way up and on the way down. As D, v_1 and a are known we have two quadratic equations in t. The solution of either one of these equations will give two values of t. These two values of t represent the time that the body is D distance above the ground.

$$s = v_1t + \frac{1}{2}at^2 \quad \left| \begin{array}{l} s = D. \\ v = 140 \text{ ft./sec.} \\ a = 32 \text{ ft./sec.}^2 \\ t = t. \end{array} \right.$$

$$D = 140t + \frac{1}{2}32t^2.$$

$$s = v_1t + \frac{1}{2}at^2 \quad \left| \begin{array}{l} s = D. \\ v = 135 \text{ ft./sec.} \\ a = 32\text{ft./sec.}^2 \\ t = (t-2) \end{array} \right.$$

$$D = 135(t-2) + \frac{1}{2}32(t-2)^2.$$

But since D is the same in both cases, we have

$$140t + \frac{1}{2}32t^2 = 135(t-2) + \frac{1}{2}32(t-2)^2$$

Solving, we have:

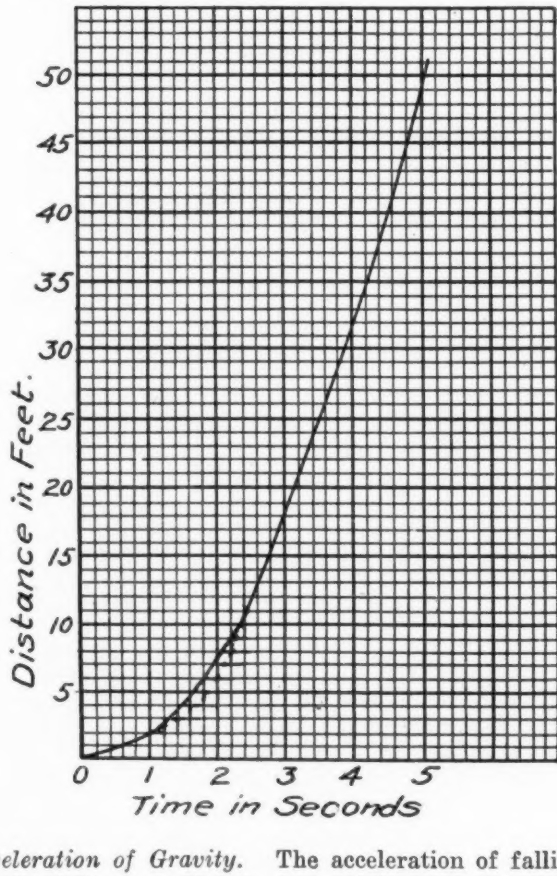
$t = 5.66$ seconds after the first projection. Ans.

If we now put for t the value just found, we will get from either equation for D the value of 278 feet. Ans.

Let us find the distances passed over each second by a body that starts from rest and has an acceleration of 4 ft./sec.^{2} The distances are 2 feet, 8 feet, 18 feet, 32 feet and 50 feet re-

spectively. If we plot distance on the ordinate and time on the abscissa, we have a space curve for uniform accelerated motion of 4 ft./sec.^{2}

The graph is a parabola.



Acceleration of Gravity. The acceleration of falling bodies at sea level is about 32.2 ft./sec., while bodies hurled upward lose velocity at this same rate. This is the reason why some baseball catchers had trouble in catching a ball dropped from Washington Monument a few years ago. The ball had acquired such a high velocity before it reached the ground.

There is a pull or attraction between any two bodies in the universe. This pull or attraction between bodies was stated

in 1685 by Newton as his "Law of Universal Gravitation" as follows: "All bodies attract each other with a force proportional to the product of their masses, and inversely proportional to the square of the distance between them." This force gives falling bodies an acceleration. This acceleration varies at different places on the earth surface. Acceleration is usually represented by g. In all our work we will take g as 32 ft./sec. or 980 cm./sec.

Now our formulas can be written as follows:

$$v = v_1 + gt.$$
$$s = v_1t + \tfrac{1}{2}gt^2.$$
$$v^2 = v^2 + 2gs.$$

If initial velocity is zero, we have

$$v = gt.$$
$$s = \tfrac{1}{2}gt^2.$$
$$v^2 = 2gs.$$

Miscellaneous Problems

1. A body starting with a velocity of 50 ft./sec. is accelerated uniformly. At the end of 5 seconds it has a velocity of 200 ft./sec. What is its acceleration? How far does it travel during the third second? Ans.: 30 ft./sec.² 155 ft.

2. A stone is thrown downward with a velocity of 96 ft/sec. and reaches the bottom of a well in three seconds. What is the depth of the well? Ans.: 432 ft.

3. A particle starting from rest describes 63 feet in the fourth second. Find the acceleration. Ans.: 18 ft./sec.²

4. The velocity of a particle changes from 10 to 25 ft./sec. in three seconds. What is the acceleration? When will its velocity be 75 ft./sec.? What is the total time of motion from rest? What space will it have passed over? Ans.: 5 ft./sec.² 10 sec. 15 sec. 562.5 ft.

5. A bullet is shot vertically upwards with an initial velocity of 1200 ft./sec. (a) How high will it ascend? (b) What is its velocity at the height of 16,000 ft.? (c) When will it reach the ground again? (d) With what velocity? (e) At what time is it 16,000 ft. above the ground? (f) Explain the double sign in (e). Ans.: (a) 4¼ M. (b) 645 ft./sec. (c) 1¼ min. (d) 1200 ft./sec. (e) 17 sec. and 58 sec.

6. Two trains, one 250, the other 420 ft. long, pass each other on parallel tracks in opposite sense, with equal velocities. A passenger in the shorter train observes that it takes the longer train just 6 sec. to pass him. What is the velocity? Ans.: 35 ft./sec.

7. A ball is thrown upward with a velocity of 75 ft./sec. When will the velocity be 22 ft./sec. and at what height will the ball be? Ans.: $80\frac{21}{64}$ ft. above the point of projection. $\frac{53}{32}$ sec. and $\frac{97}{32}$ sec. after projection.

8. The top girder of a building is x feet above the pavement. A man stands on the girder and throws a stone 144 feet upward. The stone hits the pavement seven seconds after the time of projection. Find the height of the girder above the pavement. Ans.: 112 ft.

9. If the speed of a train increases uniformly after starting for 8 minutes while the train travels 2 miles, what is the velocity acquired? Ans.: 30 M./hr.

10. A railroad train in approaching a station makes a half mile in the first, 2,000 ft. in the second, minute of its retarded motion. If the motion is uniformly retarded: (a) When will it stop? (b) What is the retardation? (c) What is the velocity at the time of retardation? (d) When will the velocity be 4 miles an hour? Ans.: (a) $4\frac{5}{8}$ min. (b) 0.18 ft./sec.^{2} (c) $49\frac{1}{3}$ ft./sec. (d) 4 min. and $4\frac{1}{2}$ sec.

11. A balloon is ascending with a uniform velocity of 28 ft./sec. and at the height of 720 feet a ball is dropped. When will the ball strike the ground and with what velocity? Ans.: 6.8 sec. 189.6 ft./sec.

12. A body is dropped into a well 84 ft. deep. How long before the sound of striking the bottom will be heard? Sound travels 1100 ft./sec. Ans.: 2.36 sec.

A BRIEF STUDY IN NON-MATHEMATICAL LOGIC

By N. J. LENNES
The Sate University of Montana, Missoula, Montana

While engaged in taking general stock of the existing literature on the teaching of mathematics the writer came again upon a paper by Ernest C. Moore entitled "Does the study of Mathematics train the mind specially or universally?" which was printed in *The Mathematics Teacher* Vol. 10 pages 1-18. A cursory reading of Moore's paper revealed certain interesting qualities which led to closer scrutiny. The purpose of the paper, as revealed by its content rather than by the title, is to show that the only reason for studying any subject is the use which the student may reasonably be expected to make in his own life of the matter actually learned. "Every form of skill that we attempt to teach him gets its place in the school program solely because he cannot live a civilized life without *practicing* it" (op. cit. p. 3). (The italics are mine).

The type of reasoning used by Moore is that so well known to students of elementary geometry which consists in enumerating all possible cases and then showing that all of these cases except one is false. We read: "Now, why should we study anything? As nearly as I can discover there are three answers which are given to this question. First, we must study subjects because we owe it to them to do so. It is a debt of honor, of reverence, of obeisance, or worship which we should pay them. . . . . The second reason for studying anything is that we cannot get along without it. It is an indispensable aid to us in doing our work. It may serve us in many ways, but we want it because in days to come we shall use it. . . . The third reason for studying certain subjects is that they perfect the mind and make it a better mind than before." (op. cit. pp. 2, 3, 4).

It is then shown, or at least the attempt is made to do so, that we have no religious obligation to study mathematics, whatever that may mean, and also that this study has no general disciplinary value, and we are left to infer that the second reason only is valid. The analogy with the method used in geometry is not perfect inasmuch as we are not informed that there is really

any valid reason at all for studying mathematics. If we assume that there is such a reason, then Moore's type of argument is exactly that described above.

In this paper we are not concerned with the general problem considered by Professor Moore but simply with the logical qualities of his argument. The first point I wish to make is that not all possible reasons for studying mathematics are enumerated in Moore's paper and that hence, as is well known by anyone who has ever been a competent student of high school geometry, the argument cannot possibly have any value. It is not surprising that one guilty of such a glaring violation of elementary canons of reasoning should be vehement in his assertion that the reasoning in geometry does not carry over to practical reasoning in ordinary life.

But what possible reasons for studying mathematics are omitted in this trilogy which is alleged to contain them all? For my purpose I need state only one. The human race in its perversity has accumulated a vast mass of information for reasons quite other than its so-called "usefulness." Money, time, and tireless energy have been spent, and are now being spent at an unprecedented rate, for the purpose of learning facts which by no stretch of the imagination can possibly be of "use" to anyone. The list is so overwhelming that one hesitates to name only a few instances. Of what use is it to know that there is iron or sodium in the sun, that the tides are caused by the moon, that Perry reached the North Pole or that there is a North Pole? We are driven to the conclusion that man, being what he is, seeks to "find out" simply for the purpose of finding out. Beyond any doubt very much of the information which has turned out to have useful corrollaries was sought and obtained without the slightest suspicion of such usefulness. It may be that this curiosity is perverse and deplorable but so long as man is so constituted that he is bound to be led by it we are not permitted to ignore it in making a list of possible reasons for studying any subject, and this is all that is required to make our point. We are therefore bound to list "desire to know" as one of the possible reasons for studying any subject. Our conclusion is then that reasonable heed to one of the cardinal principles inculcated in the study of

Elementary Geometry would have compelled the entire recasting of the paper under reviw or possibly its total abandonment.

Our next point is to note Moore's attempt to show that those who reason well on geometry reason badly on practical affairs. The time-honored investigation by Lewis (1905), with so far as I know its practically unique result, is quoted. Why does not the writer place besides this the results obtained on precisely the same point by Chambers (The Mathematics Teacher, Vol. 7, pp. 89-100), by Passano and Smith (The Mathematics Teacher, Vol. 8, pp. 79-88), and by Rogers (The Mathematics Teacher, Vol. 8, pp. 95-103)?[1] In all these investigations the results are directly opposite to that of Lewis on which the case is allowed to rest. Certainly these later investigations are as trustworthy and significant as that of Lewis. The science and art of making such tests underwent great forward strides in the ten years that intervened. If a student of high school physics should follow the example set by Professor Moore in this paper, he would certainly be the object of serious attention by his teacher. If a student of elementary mathematics should show a similar tendency not to use available evidence he would promptly be corrected. In no subject is the "will to believe" so effectively curbed as in mathematics.

If possible, even more astonishing is the interpretation which Professor Moore puts upon the well known investigation conducted by Dr. Rugg at the University of Illinois. The problem, in the language of Dr. Rugg, was "to investigate the effect of a semester's training in descriptive geometry upon specific abilities in the mental manipulation of spatial elements, (a) of a strictly geometrical type; (b) of the quasi-geometrical type; (c) of a non-geometrical type." Quoting Dr. Rugg Professor Moore says: "In 'Rights' 72.7 per cent of the training group and 31 per cent of the control group gain in 60 per cent or more of the tests taken." Professor Moore then goes on to draw his own conclusions: "If 72.7 per cent who took the training gained, we may conclude that 27 out of every hundred who took it did not gain and as 31 per cent of those who did not have it did as well as those who did have it, only 42 out of every hundred became more accurate because of it, while 58 did not, then you see the chances

[1] Note that I confine myself to investigations reported in the Mathematics Teacher at least two years before the paper under review was published.

seem to be about 6 to 4 against expecting anything in the way of general training, that is training which is not strictly specific from such a course. On Dr. Rugg's showing the dice are loaded against every student who takes this course for general training."

This is as refreshingly novel reasoning as I have ever heard in a class of college freshmen. Our premises are that 72.7 per cent of the training group gained while only 31 per cent of the control group gained and the conclusion is that "as 31 per cent of those who did not have the training did as well as those who did have it only 42 out of every hundred became more accurate because of it." Now, obviously we know nothing of the sort. We do not know that "31 per cent who did not have the training did as well as those who did have it." *How much* did the 72.7 per cent gain and *how much* did the 31 per cent gain? That is an absolutely crucial question which is not even mentioned.

Suppose we try Moore's logic on a farmer: In a dry climate the value of irrigation is to be tested. Two cornfields are carefully compared as to yield per hill of corn. The next year both fields are again planted in corn; one field is irrigated and the other is not. It is then found that 72.7 per cent of the hills in the irrigated field show a yield above the average of last year's crop while only 31 per cent of the hills in the dry field show a yield above this average. Suppose further that the average increase in the 72.7 per cent of the hills in the irrigated field is 30 per cent above the average of last year while the average increase in the 31 per cent of the dry field is only 15 per cent above the average of last year. Would the farmer believe that only 42 out of every hundred hills profited by the irrigation? But this is not the whole story. In the irrigated field 27 per cent of the hills showed a yield equal to or less than last year's average while in the non-irrigated field 69 per cent of the hills showed such a yield. How much below last year's average were these yields? Clearly this question is to the point. Fortunately Dr. Rugg's memoir contains data which though they do not enable us to answer this specific question nevertheless help us to form generally valuable conclusions. In the non-geometrical tests the average of all in the trained group showed a gain of 14.7 per cent and in the control group 7.4 per cent, in the quasi-geometrical tests these average gains were 18.5 per cent and 8.1 per cent respect-

ively and in the geometrical tests the average gain of the trained group was 43.5 per cent while the control group lost an average of 5.0 per cent. No one at all familiar with statistical matter will hesitate in saying that Moore's conclusion: "the chances seem to be about 6 to 4 against expecting anything in the way of general training," drawn as it is solely from the facts which he recites, is entirely unwarranted.

As Moore says (op. cit. p. 1) "Education is or at least aims to be a conscious process and a purposive undertaking." Quite so. In dealing with such an undertaking the kind of reasoning exemplified above is out of place. Would it not be well if those who deal with these matters were to show some of the soberness, caution, and regard for simple logical relations which are so valuable traits in one who would succeed in a simple and well ordered subject such as mathematics?

DISCUSSION

Classification of Positive Integers as Regards the Ultimate sum of Their Digits. In volume 3 of the *Oxford, Cambridge, and Dublin Messenger of Mathematics,* 1866, page 30, C. M. Ingleby noted that if N_r be any number expressed in a scale whose radix is r and if its digits be summed, and the digits of that sum be summed, and so on, it is plain that we must arrive at a sum consisting of a single digit. This sum he called the *ultimate sum* and then proved that the ultimate sum of the product of two or more positive integers is equal to the ultimate sum of the product of the ultimate sums of these integers.

Iamblichus (about 300 A. D.) had noted another special theorem relating to the ultimate sum of digits which has been quoted in the various general histories of mathematics[1], and may be stated as follows: The ultimate sum of the digits of any number which is the sum of three consecutive positive integers, of which the largest is divisible by 3, is 6. L. E. Dickson noted the article by Ingleby in volume 1 of his *History of the Theory of Numbers,* 1919, page 455, but the special earlier related work by Iamblichus is not noted there.

In view of the historical contact of these developments it may be desirable to note here a few immediate extensions which we shall express in terms of the decimal notation but they evidently apply to any base. If a positive integer n is composed of more than one digit we obtain a smaller n by adding its digits, since we thus use only one-tenth of the value of the digit in the tens place, one-hundredth of the value of the digit in the place of hundreds, etc. If n is again composed of more than one digit we obtain a still smaller number n by the operation of adding its digits, etc. Hence we must arrive at an ultimate sum n composed of a single digit by performing successively the operation of adding digits.

Hence the totality of the positive integers may be divided into 9 distinct sets, each set being composed of all those numbers whose ultimate sum is the same. This classification of all the positive integers is evidently identical with the classification according to modulus 9, the numbers whose ultimate sum is n_r

[1] Ball, A Short Account of the History of Mathematics, 5th ed., 1912, p. 110; Cajori, A History of Mathematics, 2nd ed., 1919, p. 59.

being identical with the positive integers which are congruent to n_r mod 9. The totalty of the numbers which are congruent with respect to 9 as a modulus are thus seen to constitute an invariant with respect to the operation of adding the digits when the number is expressed in the decimal form.

The three numbers involved in the theorem of Iamblichus noted above may clearly be represented as follows: $3m, + 1$, $3m + 2$, $3m + 3$. Since their sum, $9m + 6$, is congruent to 6 mod 9 the ultimate sum of this sum is 6. If the smallest of three consecutive integers is a multiple of 3 the ultimate sum of their sum is clearly 3, while $n_r = 9$ when the middle of these three consecutive numbers is a multiple of 3. These results are closely connected with the theorem of Iamblichus.

Since the m sets of positive integers obtained by classifying numbers together which are congruent with respect to m as a modulus constitute a group of order m, in the technical sense of this word, with respect to addition, it results that the 9 sets of positive integers noted above constitute a group of order 9 with respect to this operation. These sets do not constitute a group with respect to multiplication, but the six sets which involve 1, 2, 4, 5, 7, 8 respectively constitute such a group.

The main object of this note is to direct attention to the advantage of emphasizing the fact that the positive integers which are congruent with respect to 9 as a modulus constitute an invariant as regards the operating of adding digits. In particular, the theorem of Iamblichus noted above and the check on elementary operations by the method of casting out 9's are almost a direct consequence thereof. PROFESSOR G. A. MILLER,

University of Illinois.

Team Work in Elementary Algebra. When a teacher analyzes an average class of students, he can easily see that there are at least three distinct groups. He finds a small group of students of superior ability, another small group of students of inferior ability, and then the third group, which is the majority of the class, of average ability. Further, each of these groups has a gradation in itself from the high to the low. In mathematics, languages and sciences, he finds these groups quite well defined.

In schools where the classes are necessarily large, the average teacher finds it difficult to organize the work in such a way that each group is receiving the maximum instruction. Usually he has to obtain results at the sacrifice of the smaller groups. Thus the superior, as a group, is not developing to the maximnm, and the inferior is not coming up to the average. So he finds that the average efficiency of the class as a whole is not as high as it would be if the various groups were working intensively and extensively.

This problem of maximum pupil-accomplishment resulting from pupil-interest and continued effort has been on my mind for three or four years. Gradually a solution has been arrived at. Some time back it occured to me that perhaps the student would do better if he was made to feel that his mental efforts and daily work definitetly contributed to the success of some group or organization of which he was a real active member. In order then that the student would have not only a self-interest but a group interest in his algebra, I divided the class into two teams. Once the student became a member of a team, his daily performance was positive and effective to the group, for he used his time and ability to accomplish as much as he could in a normal amount of time. Naturally, such a daily attitude of the students tended to stimulate the activities of the superior members, to urge the average student to do even more, and to encourage a greater interest and accomplishment in the inferior group.

Teachers of algebra have tried the team method from time to time and have usually given it up because the teams sooner or later lost interest and dropped back to the old way. So the success of the team work depends upon the proper organization, teacher alertness to any tendencies of lagging, and daily team competition in the class with a generous opportunity for keen criticisms as to the appearance and accuracy of the work. For the past four years I have been able to establish and hold to the point of effectiveness class teams in elementary algebra. Each year the working of the teams has been keener and more successful, thus justifying the reasons for having team work.

This leads me, then, to summarize many of the values derived from team activities. If a teacher desires neat home work papers

and also good appearing board work, let him divide his class into teams. If he wishes to teach the significant value of a student being complete and accurate in arriving at his results, he should try team work. If he wishes to see the superior pupils accomplish in proportion to their ability; if he wants to see the average and lower groups do more work, let him organize his class into teams. If he wishes to arouse enthusiasm, let him use the team idea. If he wishes to correlate English with mathematics, give the teams a chance. Therefore, it is clear that team competition is an attractive and effective method for re-creating enthusiasm and sustaining interest in first year algebra classes. In short, it surely corrects or prevents many unpleasant tendencies and helps to build up a firm group and personal interest.

Team work will result in keen and active class participation.

Note: Details concerning the organization of teams and teacher activity can be had by writing Mr. Hawley.

JAMES B. HAWLEY,
Meriden High School, Meriden, Conn.

NEWS AND NOTES

Minutes of Meeting of National Council of Teachers of Mathematics Held at Cleveland, February 28, 1923

The meeting was called to order at ten o'clock by President Minnick.

The minutes of the last annual meeting were read by the Secretary and approved.

The report for the Publicity Committee was made by Mr. John R. Clark. Eight thousand circulars were printed and circulated during the past year. After discussion, it was decided, on motion, that the Publicity Committee be directed to revise the circular now in use, and cause a new edition to be printed, to be distributed with the Report of the National Committee on Mathematical Requirements.

Miss Gugle reported on the unavailing efforts made to have the Council affiliated with the Department of Superintendence. The efforts made, unsuccessfully, to have the program of the Council's meeting appear in the N. E. A. Bulletin, and in the daily press, were commented on in the course of the discussion. On motion, it was decided to hold the next annual meeting on the Saturday preceding the meeting of the Department of Superintendence.

The Secretary reported on the circulation of MATHEMATICS TEACHER, giving the distribution by States as follows:

Alabama	42	Michigan	104
Arizona	18	Minnesota	143
Arkansas	26	Mississippi	21
California	130	Missouri	68
Colorado	57	Montana	8
Connecticut	74	Nebraska	48
Delaware	14	New Hampshire	17
Washington, D.C.	22	New Jersey	138
Florida	18	New York	292
Georgia	14	New Mexico	5
Hawaii	6	Nevada	3
Idaho	15	North Carolina	55
Illinois	287	North Dakota	15
Indiana	97	Ohio	234
Iowa	64	Oklahoma	61
Kansas	102	Oregon	9
Kentucky	28	Pennsylvania	312
Louisiana	19	Phillipine Islands	2
Maine	38	Porto Rico	1
Maryland	86	Rhode Island	22
Massachusetts	264	South Carolina	26

South Dakota	22	Washington	28
Tennessee	20	West Virginia	31
Texas	75	Wisconsin	127
Utah	3	Wyoming	5
Vermont	18	Foreign	44
Virginia	34		
		TOTAL	3410

The Treasurer's report was submitted, showing a balance on hand, February 28, 1923, of $389.40. This report was referred to the Auditing Committee consisting of Dr. E. H. Taylor, F. R. Higgins, and Miss Worden.

The President announced that all the arrangements for the present meeting had been made by the Cleveland Mathematical Club.

The following named were appointed to serve as a nominating committee: Mr. Werremeyer, Dr. Walsh, Miss Hough.

Mr. Clark explained a plan proposed by President Owen of the N. E. A. for the appointment of a committee representative of all the various organizations of school people for the purpose of considering the program of studies as a whole. Each organization is requested to indicate to President Owen a list of names from which a representative of the organization may be designated. On motion, the matter of drawing up this list was left with the Executive Committee:

Professor Taylor reported for the Auditing Committee as follows:

"We, the Auditing Committee of the National Council of Teachers of Mathematics, have examined the books of the Treasurer and find them correct as stated. E. H. Taylor, F. R. Higgins, Ophelia Worden."

The Executive Committee submitted the following names to be sent President Owen from which to appoint a representative of the Council of Teachers of Mathematics: Dr. Minnick, John R. Clark, J. A. Foberg. On motion this report was accepted.

At the afternoon meeting, the following program was presented:

"Demonstrative Geometry as a Subject for Training," by D. W. Werremeyer, of the West Technical High School, Cleveland.

"Mathematics of the Junior High School," by Miss Amy Preston, of the Roosevelt Junior High School, Columbus.

"Results Obtained from the Rogers Test of Mathematical Ability," by Dr. Agnes Rogers, Goucher College, Baltimore.

At the evening banquet in the Hotel Winton, President Minnick presided. After the repast, the following papers were presented:

"Organization of the Mathematics of the Seventh, Eighth and Ninth Grades According to Pedagogic Units," by Professor E. R. Breslich, of the School of Education, University of Chicago.

"Teaching the Algebraic Language to Junior High School Pupils," by Professor J. R. Overman, Ohio State Normal School, Bowling Green, Ohio.

The following report of the nominating committee was presented:

For President: Professor J. H. Minnick, Philadelphia.

For Vice President: Miss Mabel Sykes, Chicago.

For Secretary-Treasurer: J. A. Foberg, Harrisburg.

For Members of Executive Committee: Miss Eula Weeks, St. Louis; W. C. Eels, Walla Walla.

On motion this report was accepted.

On motion, a vote of thanks was extended on behalf of the Council to the Cleveland Mathematical Club for the care taken in making arrangements for the meetings of the Council, and for the dinner.

The meeting then adjourned.

The following named people registered at the Cleveland meeting:

NAME — INSTITUTION REPRESENTED

Applegate, Eleanor, Audubon Jr. High School, Cleveland, Ohio
Arbuckle, Myrth, Patrick Henry Junior High School
Baldwin, Helen E., Empire Junior High School, Cleveland
Baxter, James H., Indiana State Normal School, Muncie, Indiana
Beckenbach, Katherine, Memphis Schools
Bell, Ernestine, Lincoln High School, Cleveland
Benson, Mary, Lincoln High School, Cleveland
Blake, L., Glenville High School, Cleveland
Breslich, E. R., Chicago, Illinois
Brown, Bertha A., Fairmount Junior High School
Burroughs Fred., E. Tech. High School, Cleveland
Clark, Blanche M., Cleveland, Ohio
Clark, John R., New York City
Clelland, H. L., Bellevue High School, Pittsburgh
Cole, Nellie E., Lincoln Junior High School, Cleveland
Cole, Ella G., Central High School, Cleveland, O.
Colman, Nellie H., Collinwood Jr. High School, Cleveland O.
Cooke, Martha, Audubon Jr. High School, Cleveland, O.
Croninger, Fred H., Central High School, Fort Wayne, Indiana
Dana, Emma K., Lincoln High School, Cleveland

DeeLackey, Jessie, East High School, Cleveland
Deland, H. L., Bellevue High School, Bellevue, Pa.
Denison, Adelaide C., Lincoln High School, Cleveland
Dickerson, Jean, Lincoln Junior High School, Cleveland
Diehl, Lulu, Audubon Jr. High School, Cleveland, O.
Domhey, Mary, Detroit Junior High School, Cleveland
Durham, E. B., Sandusky High School, Sandusky, Ohio
Eykyn, Laura M., Patrick Henry Junior High School
Fardy, Katherine, West Junior High School
Formanek, Anna C., Kennard Junior High School, Cleveland
Gage, Sarah, Audubon Junior High School
Gehlke, Helen, Empire Jr. High School, Cleveland, O.
Gerber, A. J., West High School, Akron, O.
Giltman, Ellen M., E. Tech. High School, Cleveland
Glenn, Frances, South High School, Cleveland
Groat, Agnes, Kennard Jr. High School, Cleveland, O.
Groot, Isabelle D., Addison Junior High School, Cleveland
Gugle, Mario, Asst. Supt. Columbus, Ohio
Haber, H. F., East High School, Cleveland
Hazel, H. R., Glenville High School, Cleveland
Herold, Gladys, Audubon Jr. High School, Cleveland, Ohio
Higgins, Frank R., Indiana State Normal School, Terre Haute, Indiana
Hitchcock, A. H., Central High School, Cleveland, O.
Hoffman, Alice P., Gilbert Junior High School, Cleveland
Huff, Louise H., (Secretary and treasurer of St. Louis Math. Club) Cleveland, Ohio, High School, St. Louis, Mo.
Jacobs, J. M., Glenville High School, Cleveland
Keegan, Sara U., Detroit Junior High School, Cleveland
Kelley, M. G., Brownell Jr. High School, Cleveland, O.
Kennedy, H. W., E. Tech. High School, Cleveland
Kephart, R. C., South High School, Cleveland
Kerr, George P., Lincoln High School, Cleveland
Klaustermeyer, Carol, Collinwood Jr. High School, Cleveland, O.
Knights, Ethel, E. Tech. High School, Cleveland
Kohnky, Frances, Walnut Hills Classical School, Cincinnatti, Ohio
Kraft, Ova, East High School, Cleveland
Lawrence, Dean, E. Tech. High School, Cleveland
Lindesmith, W. B., West Tech. High School, Cleveland
Marshall, L. M., Brownell High School
Marshall, Mrs. L. M., Brownell Junior High School
Malley, Mary O., Lincoln High School, Cleveland
McCabe, Helen, Audubon Jr. High School, Cleveland, Ohio
McQuay, L. E., Rawlings Jr. High School
Medlin, Johanna W., West Junior High School
Miller, Florence B., Fairmount Junior High School, Cleveland
Miller, Josephine, Lincoln Junior High School, Cleveland
Miller, A. Brown, Fairmount Junior High School, Cleveland
Mirick, Gordon R., The Scarborough School, Scarborough-on-Hudson, N.Y.
Morris, W. W., East High School, Cleveland
Morrow, Katherine O., Central High School, Cleveland, O.
Munhall, Helen, Detroit Junior High School, Cleveland
McCarthy, Anne G., West Junior High School
McGowan, Laura T., E. Tech. High School, Cleveland
McMeen, Edith, Addison Junior High School, Cleveland
Nichols, Cora Kendall, Patrick Henry Junior High School
O'Brien, Adelaide, Central High School, Cleveland, O.
Overman, J. R., Bowling Green, Ohio
Petty, Ruth E., Empire Junior High School, Cleveland

Petersilge, Arthur F. M., Cleveland, O.
Philbrick, Metta G., Pilgrim Jr. High School, Columbus, Ohio
Pitcher, A. D., Western Reserve University, Cleveland, O.
Powell, Mrs. Maggie, Kennard Junior High School, Cleveland
Preston, Amy F., Roosevelt Junior High School, Columbus
Radcliffe, Eleanor J., Brownell Jr. High School, Cleveland, O.
Rader, Madge, Junior High School, Cleveland
Radke, Ruth M., Rawlings Jr. High, Cleveland, O.
Rahe, Lucia K., Kennard Junior High School, Cleveland
Rauch, Annamary, Cleveland, Ohio
Reiff, Marie, Sandusky High School, Sandusky, Ohio
Rike, E. R., Woodward Tech., Toledo, Ohio
Rolli, D. C., Lincoln High School, Cleveland
Roberts, Gertrude, Junior High School, Huntington, W. Va.
Rogers, Agnes L., Goucher College, Baltimore, Maryland
Rowlands, Grace E., Junior High School, Cleveland
Rudin, Melanae, Brownell Junior High School, Cleveland
Sampson, Helen W., E. Tech. High School, Cleveland
Sapp, Mrs. Netta, E. Tech. High School, Cleveland
Saarbach, Julia O., Rawlings Jr. High School, Cleveland, O.
Schaefer, A. J., South Junior and Senior High School
Schorling, Raleigh, The Lincoln School, New York City
Scott, Flora L., Audubon Junior High School
Sechrist, Edith, E. Tech. High School, Cleveland
Secor, Elizabeth, Collinwood Jr. H. S., Cleveland O.
Shellenbarger, R. C., Turner College, Bay City, Michigan
Shriner, Walter O., Shaker Heights Schools, Ohio
Shwely, Helen E., Addison Junior High School
Simpson, Helen D., Lorain High School, Lorain, Ohio
Simon, Webster, Western Reserve University High
Shellenbarger, R. C., Junior College, Bay City, Mich.
Spiers, Nelle R., Empire Junior High School, Cleveland
Stroup, P., West High School, Cleveland
Swartzel, K. D., University of Pittsburgh
Taylor, E. H., State Teachers College, Charleston, Ill.
Taylor, J. F., Denfield High School, Duluth, Minn.
Terry, Florence, Addison Junior High School
Thomas, Elizabeth J., Patrick Henry Jr. High School, Cleveland, O.
Titus, Harriet, Empire Junior High School, Cleveland
Tremper, C. B., E. Tech. High School, Cleveland
Walsh, C. B., Woodmere Academy, Woodmere, N. Y.
Walsh, Margaret M., Willoughby High School, Willoughby, Ohio
Weber, Minnie A., Patrick Henry Junior High School
Weigner, B. H., Empire Junior High School
Welbaum, C. R., South High School, Akron, O.
Werremeyer, D. W., West Technical High School, Cleveland, O.
Whelan, Cecily, Brownell Junior High School
Wilder, Louise B., Brownell Jr. High School, Cleveland, O.
Wright, S. M., Rawlings Jr. High School, Cleveland, Ohio
Wright, L. M., Kennard Jr. High School, Cleveland, O.
Worden, Orpha E., Detroit Teachers College, Detroit, Mich.
Zook, D. B., S. High School, Akron, Ohio
Zucker, Vevia R., Audubon Jr. High School, Cleveland, Ohio